Unit - I

Field Theory.

Gradient - Directional Derivative.

1) The vector differential operator ∇

The vector differential operator ∇ (del) is defined as

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes OX, OY and OZ.

2) The Gradient [or slope of a scalar point function].

Let $\phi(x,y,z)$ be a scalar point function and continuously differentiable. Then the vector

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

is called the gradient of the scalar point function ϕ.

$\therefore$ Grad $\phi = \nabla\phi$.

Note:

* ∇ is a vector differential operator
* The components of $\nabla\phi$ are
$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$
* If ϕ is a constant, then $\nabla\phi = \vec{0}$.
* $\nabla(c_1\phi_1 \pm c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$

c_1 and c_2 are constants.

ϕ_1, ϕ_2 are scalar point functions

* $\nabla(f \pm g) = \nabla f \pm \nabla g$
* $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$
* $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$

if $\phi_2 \neq 0$.

* If $V = f(u)$, then

$\nabla V = f'(u) . \nabla u$.

* $\nabla r \equiv \sum \vec{i}\, \frac{\partial}{\partial x}(r)$

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Problems:

① Find $\nabla(r)$ & $\nabla\left(\frac{1}{r}\right)$.

16m / r(1)

Soln we know that

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}.$$

$$\therefore r^2 = x^2+y^2+z^2.$$

Diff w.r.x, y, z, we get

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

w.r. 'y'

$$2r \cdot \frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

with respect to 'z'

$$2r \cdot \frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}.$$

(1) $\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$.

$$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} = \frac{\vec{r}}{r}.$$

(2) Find $\nabla\left(\frac{1}{r}\right)$

we know that

$$\nabla r = \Sigma \vec{i}\frac{\partial}{\partial x}(r)$$

$$\therefore \nabla\left(\frac{1}{r}\right) = \Sigma \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \Sigma \vec{i}\left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}$$

$$= \Sigma \vec{i}\left(-\frac{1}{r^2}\right) \cdot \frac{x}{r}$$

$$= \Sigma \vec{i}\left(-\frac{x}{r^3}\right)$$

$$= -\frac{1}{r^3} \cdot \Sigma x\vec{i}.$$

$$= -\frac{1}{r^3} \cdot [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}.$$

③. If $\phi = xyz$, then find $\nabla\phi$.

Given, $\phi = xyz$ — ①

w.k.t $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ — ②

Diff ①, w.r.'x'

$\frac{\partial\phi}{\partial x} = yz$

Diff ① w.r.'y'

$\frac{\partial\phi}{\partial y} = xz$.

Diff ①, w.r, 'z'

$\frac{\partial\phi}{\partial z} = xy$.

Substitute all these values in eqn ②,

we get

$\nabla\phi = \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$

$= yz\vec{i} + xz\vec{j} + xy\vec{k}$.

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③ If $\phi = \log(x^2+y^2+z^2)$, then find $\nabla\phi$.

Given $\phi = \log(x^2+y^2+z^2)$.

$\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x)$. $\quad d(\log x) = \frac{1}{x}$

$\frac{\partial\phi}{\partial y} = \frac{1}{x^2+y^2+z^2}(2y)$.

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$$\frac{\partial \phi}{\partial z} = \frac{1}{x^2+y^2+z^2} \cdot (2z)$$

then $\nabla \phi = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z}$

$$= \vec{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right)$$

$$+ \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}\left[x\vec{i}+y\vec{j}+z\vec{k}\right]$$

$$= \frac{2 \cdot \vec{r}}{x^2+y^2+z^2} = \frac{2 \cdot \vec{r}}{r^2}$$

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(4). Find $\nabla(\log r)$.

$$\nabla(\log r) = \sum \vec{i}\,\frac{\partial}{\partial x}(\log r)$$

$$= \sum \vec{i}\,\frac{1}{r}\cdot\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\cdot\frac{1}{r}\cdot\left(\frac{x}{r}\right)$$

$$= \sum \vec{i}\cdot\frac{x}{r^2} = \frac{1}{r^2}\cdot\sum x\vec{i}$$

$$= \frac{1}{r^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{\vec{r}}{r^2}$$

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(5) Prove that $\nabla(r^n) = n r^{n-2} \cdot \vec{r}$

16m **Soln**

$$\nabla(r^n) = \sum \vec{i} \frac{\partial}{\partial x}(r^n)$$

$$= \sum \vec{i}\ n r^{n-1} \cdot \frac{\partial r}{\partial x}$$

$$= \sum \vec{i} \cdot n r^{n-1} \cdot \frac{x}{r}$$

$$= \sum \vec{i} \cdot n r^{n-1} \cdot x \cdot r^{-1}$$

$$= \sum \vec{i} \cdot n r^{n-2} \cdot x$$

$$= n \cdot r^{n-2} \cdot \sum x \vec{i}$$

$$= n \cdot r^{n-2} \cdot \vec{r}.$$

$$\boxed{\nabla(r^n) = n \cdot r^{n-2} \cdot \vec{r}}$$

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(6) prove that $\nabla f(r) = \frac{f'(r)}{r} \cdot \vec{r}$.

16m where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\nabla f(r) = \sum \vec{i} \frac{\partial}{\partial x} f(r)$$

$$= \sum \vec{i} \cdot f'(r) \cdot \frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\ f'(r) \cdot \frac{x}{r}.$$

$$= \frac{f'(r)}{r} \cdot \sum x\vec{i} = \frac{f'(r)}{r} \vec{r}.$$

⑦. prove that grad $(\phi\psi) = \phi$ grad $\psi +$
ψ grad ϕ.

<u>proof:</u>

$$\text{grad}(\phi\psi) = \nabla(\phi\psi)$$

$$= \sum \vec{i} \frac{\partial}{\partial x}(\phi\psi)$$

$$= \sum \vec{i} \left[\phi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \phi}{\partial x}\right]$$

$$= \sum \vec{i} \left(\phi \frac{\partial \psi}{\partial x}\right) + \sum \vec{i} \left(\psi \frac{\partial \phi}{\partial x}\right)$$

$$= \phi \cdot \left(\sum \vec{i} \frac{\partial \psi}{\partial x}\right) + \psi \left(\sum \vec{i} \frac{\partial \phi}{\partial x}\right)$$

$$= \phi \cdot \nabla \psi + \psi \cdot \nabla \phi.$$

$$\text{grad}(\phi\psi) = \phi \text{ grad } \psi + \psi \cdot \text{grad } \phi$$

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⑧. prove that $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi \nabla \phi - \phi \nabla \psi}{\psi^2}$ $(\psi \neq 0)$

<u>proof:</u>

$$\nabla\left(\frac{\phi}{\psi}\right) = \sum \vec{i} \frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right)$$

$$= \sum \vec{i} \left[\frac{\psi \frac{\partial \phi}{\partial x} - \phi \cdot \frac{\partial \psi}{\partial x}}{\psi^2}\right]$$

$$= \frac{1}{\psi^2}\left[\Sigma \vec{i}\left(\psi \frac{\partial \phi}{\partial x} - \phi \frac{\partial \psi}{\partial x}\right)\right]$$

$$= \frac{1}{\psi^2}\left[\psi\left(\Sigma \vec{i} \frac{\partial \phi}{\partial x}\right) - \phi \cdot \left(\Sigma \vec{i} \frac{\partial \psi}{\partial x}\right)\right]$$

$$= \frac{1}{\psi^2}\left[\psi\left(\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}\right) - \phi\left(\vec{i} \frac{\partial \psi}{\partial x} + \vec{j} \frac{\partial \psi}{\partial y} + \vec{k} \frac{\partial \psi}{\partial z}\right)\right]$$

$$\nabla\left(\frac{\phi}{\psi}\right) = \frac{1}{\psi^2}\left[\psi \cdot \nabla \phi - \phi \cdot \nabla \psi\right]$$

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(9) P.T Gradient of a constant is a null vector.

Soln If $\phi(x, y, z)$ is a constant.

we prove, Gradient of of a constant is a null vector.

If $\phi(x, y, z) = k$ (constant)

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 0.$$

$$\therefore \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0)$$

$$= \vec{0}.$$

PT $\nabla(e^{x^2+y^2+z^2}) = 2e^{r^2}\cdot\vec{r}$

Soln: We know that

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla(e^{x^2+y^2+z^2}) = \nabla(e^{r^2})$$

$$= \Sigma \vec{i}\, \frac{\partial}{\partial x}(e^{r^2})$$

$d(e^x) = e^x$

$\frac{\partial}{\partial x}(e^{r^2}) = e^{r^2}\cdot 2r\frac{\partial r}{\partial x}$

$$= \Sigma \vec{i}\, e^{r^2}\cdot 2r\cdot \frac{\partial r}{\partial x}$$

$$= \Sigma \vec{i}\, e^{r^2}\cdot 2r\cdot \frac{x}{r}$$

$$= \Sigma \vec{i}\, e^{r^2} \times 2x$$

$$= 2e^{r^2}\left[\Sigma x\vec{i}\right]$$

$$= 2e^{r^2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= 2e^{r^2}\cdot\vec{r}$$

Directional Derivative

$$\text{Directional Derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

Problems:

(1). Find the Directional Derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of (i) $2\vec{i} - \vec{j} - 2\vec{k}$

(ii) $2\vec{i} + 3\vec{j} + 4\vec{k}$.

Soln. Given $\phi = x^2yz + 4xz^2$ — (1)

we know that

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \quad \text{— (2)}$$

Diff. (1) ⟹ with respect to x

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2.$$

$$\frac{\partial\phi}{\partial y} = x^2z$$

$$\frac{\partial\phi}{\partial z} = x^2y + 4x \cdot 2z = x^2y + 8xz,$$

Substitute all these values in (2)

$$\nabla\phi = \vec{i}\,(2xyz + 4z^2) + \vec{j}\,(x^2z) + \vec{k}\,(x^2y + 8xz). \quad \text{— (3)}$$

$\nabla\phi$ at $(x=1, y=-2, z=-1)$.

sub $x=1, y=-2, z=-1$ in (B)

we get

$\nabla\phi(1,-2,-1) = [2 \times 1 \times (-2) \times (-1) + 4(-1)^2]\vec{i} + [1^2 \times (-1)]\vec{j} + [1^2 \times (-2) + 8 \times 1 \times (-1)]\vec{k}$

$= 8\vec{i} - \vec{j} - 10\vec{k}$.

(i) $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$.

$|a| = \sqrt{2^2 + (-1)^2 + (-2)^2}$

$= \sqrt{4+1+4} = \sqrt{9} = 3$.

Directional Derivative $= \nabla\phi * \frac{\vec{a}}{|\vec{a}|}$

$= (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$

$i \cdot i = 1$
$j \cdot j = 1$
$k \cdot k = 1$

$= \frac{16 + 1 + 20}{3} = \frac{37}{3}$

$i \times j = 0$
$j \times k = 0$
$i \times k = 0$

(ii) $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$|a| = \sqrt{2^2 + 3^2 + 4^2}$

$= \sqrt{4+9+16} = \sqrt{29}$

$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$= (8\vec{i}-\vec{j}-10\vec{k}) \frac{(2\vec{i}+3\vec{j}+4\vec{k})}{\sqrt{29}}$

$= \frac{16-3-40}{\sqrt{29}} = \frac{-27}{\sqrt{29}}$

② Find the directional derivative of $4x^2z + xy^2z$ at $(1,-1,2)$ in the direction of $2\vec{i}-\vec{j}+3\vec{k}$

Soln given $\phi = 4x^2z + xy^2z$ —— ①

we know that

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ —— ②

Diff ①, w.r. 'x'

$\frac{\partial\phi}{\partial x} = 8xz + y^2z$

Diff ①, w.r. 'y'

$\frac{\partial\phi}{\partial y} = 2xyz$

Diff ①, w.r. 'z'

$\frac{\partial\phi}{\partial z} = 4x^2 + xy^2$

Substitute all these values in ②

$$\nabla\phi = \vec{i}\,(8xz + y^2 z) + \vec{j}\,(2xyz) + \vec{k}\,(4x^2 + xy^2)$$

$$\nabla\phi\,(1,-1,2) = \vec{i}\,[8 \times 1 \times 2 + (-1)^2 \times 2) + \vec{j}\,(2 \times (1) \times (-1) \times 2) + \vec{k}\,[4 \times 1^2 + 1 \times (-1)^2]$$

$$= (16+2)\,\vec{i} + (-4)\,\vec{j} + (4+1)\,\vec{k}$$

$$\nabla\phi = 18\vec{i} - 4\vec{j} + 5\vec{k}.$$

given $\vec{a} = 2\vec{i} - \vec{j} + 3\vec{k}.$

$$|a| = \sqrt{2^2 + (-1)^2 + 3^2}$$

$$= \sqrt{4+1+9} = \sqrt{14}.$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (18\vec{i} - 4\vec{j} + 5\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} + 3\vec{k})}{\sqrt{14}}$$

$$= \frac{36+4+15}{\sqrt{14}} = \frac{55}{\sqrt{14}}.$$

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(3) Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1, 2, 3)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$

<u>Soln</u> Given $\phi = x^2yz + 4xz^2 + xyz$ — ①

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \quad \text{— ②}$$

① ⇒ $$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2 + yz$$

$$\frac{\partial\phi}{\partial y} = x^2z + xz$$

$$\frac{\partial\phi}{\partial z} = x^2y + 8xz + xy$$

② ⇒ $$\nabla\phi = \vec{i}(2xyz + 4z^2 + yz) + \vec{j}(x^2z + xz) + \vec{k}(x^2y + 8xz + xy)$$

$$\nabla\phi(1,2,3) = \vec{i}[(2\times1\times2\times3) + 4\times3^2 + 2\times3)] + \vec{j}[1^2\times3 + 1\times3] + \vec{k}[1^2\times2 + 8\times1\times3 + 1\times2]$$

$= \vec{i}\,[12+36+6] + \vec{j}\,[3+3]$
$+ \vec{k}\,[2+24+2]$

$= \vec{i}\,54 + \vec{j}\,6 + \vec{k}\,28$

$= 54\vec{i} + 6\vec{j} + 28\vec{k}$

Given $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$\therefore |a| = \sqrt{2^2 + 1^2 + (-1)^2}$

$= \sqrt{4+1+1} = \sqrt{6}$.

$\therefore D.D = \nabla\phi \cdot \dfrac{\vec{a}}{|a|}$

$= (54\vec{i} + 6\vec{j} + 28\vec{k}) \cdot \dfrac{(2\vec{i} + \vec{j} - \vec{k})}{\sqrt{6}}$

$= \dfrac{108 + 6 - 28}{\sqrt{6}} = \dfrac{86}{\sqrt{6}}$.

(4) In what direction from the point $(3, 1, -2)$ is the directional derivative of $\phi = x^2 y^2 z^4$ a maximum? What is the magnitude of this maximum?

Soln $\phi = x^2 y^2 z^4$.

$$\frac{\partial\phi}{\partial x} = 2xy^2z^4 = 2xy^2z^4$$

$$\frac{\partial\phi}{\partial y} = x^2 2yz^4 = 2x^2yz^4$$

$$\frac{\partial\phi}{\partial z} = x^2y^2 4z^3 = 4x^2y^2z^3$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}.$$

$$= \vec{i}(2xy^2z^4) + \vec{j}(2x^2yz^4) + \vec{k}(4x^2y^2z^3)$$

$$= 2xy^2z^4\vec{i} + 2x^2yz^4\vec{j} + 4x^2y^2z^3\vec{k}$$

$$\nabla\phi(3,1,-2) = 2\times3\times1^2\times(-2)^4\vec{i} + 2\times3^2\times1\times(-2)^4\vec{j} + 4\times3^2\times1^2\times(-2)^3\vec{k}$$

$$\nabla\phi = 96\vec{i} + 288\vec{j} - 288\vec{k}.$$

$$|\nabla\phi| = \sqrt{(96)^2 + (288)^2 + (-288)^2}$$

$= \sqrt{9216 + 82944 + 82944}$

$|\nabla\phi| = \sqrt{175104}$

The directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is $|\nabla\phi| = \sqrt{175104}$.

(5). Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $p(1, -2, -1)$. (i) that is maximum (ii) in the direction of PQ, where Q is $(3, -3, -2)$.

Soln Given $\phi = x^2yz + 4xz^2$.

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2$$

$$\frac{\partial\phi}{\partial y} = x^2z.$$

$$\frac{\partial\phi}{\partial z} = x^2y + 8xz$$

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$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= (2xyz + 4z^2)\vec{i} + (x^2 z)\vec{j} + (x^2 y + 8xz)\vec{k}$$

$$\therefore \nabla\phi(1,-2,-1) = [2\times1\times(-1)\times-2 + 4\times(-1)^2]\vec{i} + [(1^2\times(-1)]\vec{j} + [1^2\times(-2) + 8\times1\times(-1)]\vec{k}$$

$$\nabla\phi = 8\vec{i} - \vec{j} - 10\vec{k}$$

The magnitude of $(\nabla\phi)_P$ is the greatest directional derivative of ϕ at P.

Thus the maximum directional derivative of ϕ at $(1,-2,-1)$

$$|\nabla\phi| = \sqrt{8^2 + (-1)^2 + (-10)^2}$$

$$= \sqrt{64+1+100} = \sqrt{165} \text{ units}$$

We want find, the directional derivative of ϕ in the direction of $\overrightarrow{PQ}$

$\vec{PQ} = \vec{OQ} - \vec{OP}$

$\vec{OQ} = 3\vec{i} - 3\vec{j} - 2\vec{k}$

$\vec{OP} = \vec{i} - 2\vec{j} - \vec{k}$

$\therefore \vec{PQ} = (3\vec{i} - 3\vec{j} - 2\vec{k}) - (\vec{i} - 2\vec{j} - \vec{k})$

$= 2\vec{i} - \vec{j} - \vec{k}$

$\therefore$ D.D of $\vec{PQ} = \nabla\phi \cdot \dfrac{\vec{PQ}}{|\vec{PQ}|}$

$= (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \dfrac{(2\vec{i} - \vec{j} - \vec{k})}{\sqrt{2^2 + 1^2 + (-1)^2}}$

$= \dfrac{16 + 1 + 10}{\sqrt{4+1+1}} = \dfrac{27}{\sqrt{6}}$ units.

b. Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point $(-1, 2, 1)$.

Soln Given $\phi = xy^2 + yz^2$ — (1)

$\frac{\partial \phi}{\partial x} = y^2$, $\frac{\partial \phi}{\partial y} = 2xy + z^3$

$\frac{\partial \phi}{\partial z} = 3yz^2$.

The equation of the surface $x \log z - y^2 + 4 = 0$ is identified with $\psi(x, y, z) = c$ — ②

$\therefore x \log z - y^2 = -4$ — ③

compare ② & ③

$\psi(x, y, z) = x \log z - y^2$

$c = -4$.

$\therefore \psi = x \log z - y^2$. — ④

Diff ④ w.r.t. x, y, z

$\frac{\partial \psi}{\partial x} = \log z$

$\frac{\partial \psi}{\partial y} = -2y$

$\frac{\partial \psi}{\partial z} = x \cdot \frac{1}{z}$.

$$\therefore \nabla\psi = \frac{\partial\psi}{\partial x}\vec{i} + \frac{\partial\psi}{\partial y}\vec{j} + \frac{\partial\psi}{\partial z}\vec{k}$$

$$= (\log z)\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}.$$

$$\therefore \nabla\psi(-1, 2, 1) = \log(-1)\vec{i} - 2\times 2\vec{j} - \frac{1}{1}\vec{k}$$

$$\nabla\psi. = -4\vec{j} - \vec{k} = \vec{b}\text{(say)}$$ $\boxed{\log(1) = 0}$

then,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}.$$

$$\nabla\phi(2,-1,1) = (-1)^2\vec{i} + [(2\times 2\times(-1) + 1^3]\vec{j} + 3\times 2\times(1^2)\vec{k}$$

$$= \vec{i} - 3\vec{j} - 3\vec{k}.$$

Directional derivative of ϕ in the direction of $\vec{b}$ $= \nabla\phi . \frac{\vec{b}}{|\vec{b}|}$.

$$\vec{b} = -4\vec{j} - \vec{k}$$

$$|b| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{16+1} = 17$$

$\therefore \vec{b} = (\vec{i} - 3\vec{j} - 3\vec{k}) \frac{(-4\vec{j} - \vec{k})}{\sqrt{17}}$

$= \frac{0 + 12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$ units.

$i \cdot i = 1$
$j \cdot j = 1$
$k \cdot k = 1$

Unit tangent vector.

Unit tangent vector $= \frac{d\vec{r}/dt}{\left|\frac{d\vec{r}}{dt}\right|}$

① Find a unit tangent vector to the following surfaces at the specified points $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$ at $t = 2$

16m

Soln $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ — ①

given, $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$

① ⇒ $\vec{r} = (t^2 + 1)\vec{i} + (4t - 3)\vec{j} + (2t^2 - 6t)\vec{k}$

$\frac{d\vec{r}}{dt} = 2t\,\vec{i} + 4\vec{j} + (4t - 6)\vec{k}$

$\left(\frac{d\vec{r}}{dt}\right)_{t=2} = 2\times2\,\vec{i} + 4\vec{j} + (4\times2 - 6)\vec{k}$

$= 4\vec{i} + 4\vec{j} + 2\vec{k}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4^2 + 4^2 + 2^2} = \sqrt{16 + 16 + 4}$

$= \sqrt{36} = 6.$

$$\text{Unit tangent vector} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$$

$$= \frac{4\vec{i}+4\vec{j}+2\vec{k}}{6}$$

$$= \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}$$

2. Find the unit tangent vector to the curve $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2-65)\vec{k}$ at $t=1$.

16ma

Soln

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Given $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2-65)\vec{k}$.

$$\frac{d\vec{r}}{dt} = (2t)\vec{i} + 4\vec{j} + 4t\vec{k}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=1} = 2\vec{i} + 4\vec{j} + 4\vec{k}.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{2^2+4^2+4^2}$$

$$= \sqrt{4+16+16} = \sqrt{36} = 6.$$

unit tangent vector $= \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

$= \frac{2\vec{i}+4\vec{j}+4\vec{k}}{6}$

$= \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$

Normal Derivative

Normal Derivative $= |\nabla\phi|$.

(1) Find the normal derivative of $\phi = xy+yz+zx$ at $(-1,1,1)$.

Soln Given, $\phi = xy+yz+zx$.

we know that

$\nabla\phi = \sum \vec{i}\,\frac{\partial\phi}{\partial x}$

$= \sum \vec{i}\,\frac{\partial(xy+yz+zx)}{\partial x}$

$= \sum \vec{i}\,[y+z]$.

$\nabla\phi = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$

$(\nabla\phi)_{(-1,1,1)} = (1+1)\vec{i} + (-1+1)\vec{j} + (-1+1)\vec{k}$

$= 2\vec{i} + 0\vec{j} + 0\vec{k}$.

Normal Derivative $= |\nabla\phi|$

$= \sqrt{2^2 + 0^2 + 0^2} = \sqrt{4} = 2$

$$\boxed{|\nabla\phi| = 2}$$

Q) what is the greatest rate of increase of $\phi = xyz^2$ at $(1, 0, 3)$.

given $\phi = xyz^2$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

(or) $\nabla\phi = \Sigma\vec{i}\frac{\partial\phi}{\partial x}$.

$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$.

$\nabla\phi\,(1,0,3) = \vec{i}(0) + \vec{j}(1\times 3^2) + \vec{k}(2\times1\times0\times3)$

$= 9\vec{j}$

$\therefore$ Greatest rate of increase

$= |\nabla\phi| = \sqrt{9^2} = 9$.

$$\boxed{|\nabla\phi| = 9}$$

Unit normal vector

Unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$.

① Find the unit normal to the surface $xy = z^2$ at the point $(1,1,-1)$.

Given $xy = z^2$

$\therefore \phi = xy - z^2$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \quad \text{—— ①}$$

$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x, \quad \frac{\partial\phi}{\partial z} = -2z$

$\therefore ① \Rightarrow \nabla\phi = \vec{i}y + \vec{j}x - \vec{k}2z$

$(\nabla\phi)_{(1,1,-1)} = \vec{i} + \vec{j} + 2\vec{k}$

$|\nabla\phi| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$.

$$\boxed{|\nabla\phi| = \sqrt{6}}$$

Unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{6}}.$$

(2). Find the unit normal vector to the surface x^2+xy+y^2+xyz at $(1,-2,1)$

Given $\phi = x^2+xy+y^2+xyz$.

$$\frac{\partial \phi}{\partial x} = 2x+y+yz.$$

$$\frac{\partial \phi}{\partial y} = x+2y+xz$$

$$\frac{\partial \phi}{\partial z} = xy.$$

$$\therefore \nabla\phi = \vec{i}\frac{\partial \phi}{\partial x}+\vec{j}\frac{\partial \phi}{\partial y}+\vec{k}\frac{\partial \phi}{\partial z}.$$

$$= \vec{i}(2x+y+yz)+\vec{j}(x+2y+xz) + \vec{k}(xy)$$

$$\nabla\phi(1,-2,1) = \vec{i}(2-2-2)+ \vec{j}(1-4+1)+\vec{k}(1\times-2)$$

$$= -2\vec{i}-2\vec{j}-2\vec{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2+(-2)^2+(-2)^2}$$

$$= \sqrt{4+4+4} = \sqrt{12} = \sqrt{4\times3}$$

$$= \sqrt{4}\sqrt{3} = 2\sqrt{3}.$$

unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-2\vec{i}-2\vec{j}-2\vec{k}}{2\sqrt{3}}$$

$$\hat{n} = -\frac{(\vec{i}+\vec{j}+\vec{k})}{\sqrt{3}}.$$

③ Find the unit normal vector to the given surface $x^2y+2xz=4$ at $(2,-2,3)$.

given $\phi = x^2y+2xz-4$.

$$\frac{\partial\phi}{\partial x} = 2xy+2z$$

$$\frac{\partial\phi}{\partial y} = x^2$$

$$\frac{\partial\phi}{\partial z} = 2x.$$

$$\therefore \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2xy+2z) + \vec{j}(x^2) + \vec{k}(2x)$$

$$\nabla\phi(2,-2,3) = \vec{i}(2\times2\times-2+2\times3)$$
$$+\vec{j}(2^2) + \vec{k}(2\times2).$$

$$= \vec{i}(-8+6) + 4\vec{j} + 4\vec{k}$$

$$\nabla\phi = -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2 + 4^2 + 4^2}$$

$$= \sqrt{4+16+16}$$

$$= \sqrt{36} = 6.$$

unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{-\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$= \frac{\vec{i} - 2\vec{j} - 2\vec{k}}{3}$$

(4). Find the unit normal vector to the given surface $x^2+y^2+2z^2=26$ at the point $(2,2,3)$.

Given $\phi = x^2+y^2+2z^2-26$

$\frac{\partial\phi}{\partial x} = 2x$ | $\frac{\partial\phi}{\partial y} = 2y$ | $\frac{\partial\phi}{\partial z} = 4z$.

$$\therefore \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(4z).$$

$$(\nabla\phi)_{(2,2,3)} = \vec{i}(2\times2) + \vec{j}(2\times2) + \vec{k}(4\times3)$$

$$= 4\vec{i} + 4\vec{j} + 12\vec{k}.$$

$$|\nabla\phi| = \sqrt{4^2 + 4^2 + 12^2}$$

$$= \sqrt{16+16+144} = \sqrt{176}.$$

unit normal vector $= \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$.

$$= \frac{4\vec{i} + 4\vec{j} + 12\vec{k}}{\sqrt{176}}.$$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}.$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}\right]$$

① Find the angle between the surfaces $z = x^2+y^2-3$ and $x^2+y^2+z^2=9$ at $(2,-1,2)$.

16

given, $\phi_1 = x^2+y^2-z-3$

$\phi_2 = x^2+y^2+z^2-9$.

$\frac{\partial \phi_1}{\partial x} = 2x$ | $\frac{\partial \phi_2}{\partial x} = 2x$

$\frac{\partial \phi_1}{\partial y} = 2y$ | $\frac{\partial \phi_2}{\partial y} = 2y$

$\frac{\partial \phi_1}{\partial z} = -1$ | $\frac{\partial \phi_2}{\partial z} = 2z$.

we know that

$$\nabla\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$\therefore \nabla\phi_1 = \vec{i}\frac{\partial \phi_1}{\partial x} + \vec{j}\frac{\partial \phi_1}{\partial y} + \vec{k}\frac{\partial \phi_1}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1)$$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$(\nabla\phi_1)(2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}.$$

$$|\nabla\phi_1| = \sqrt{4^2+(-2)^2+(-1)^2}$$

$$= \sqrt{16+4+1} = \sqrt{21}.$$

lly

$$\nabla\phi_2 = \vec{i}\,\frac{\partial\phi_2}{\partial x} + \vec{j}\,\frac{\partial\phi_2}{\partial y} + \vec{k}\,\frac{\partial\phi_2}{\partial z}.$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$(\nabla\phi_2)(2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}.$$

$$|\nabla\phi_2| = \sqrt{4^2+(-2)^2+4^2}$$

$$= \sqrt{16+4+16} = \sqrt{36} = 6.$$

The angle between the surfaces

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$= \frac{(4\vec{i} - 2\vec{j} - \vec{k})(4\vec{i} - 2\vec{j} + 4\vec{k})}{\sqrt{21} \cdot 6}$$

$$= \frac{16+4-4}{\sqrt{21}\times 6} = \frac{\cancel{16}^{\,8}}{\sqrt{21}\times\cancel{6}_{\,3}} = \frac{8}{3\sqrt{21}}.$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

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(2) Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point $(6, 4, 3)$.

16m ✓

given, $\phi_1 = x^2 - y^2 - z^2 - 11$

$\phi_2 = xy + yz - zx - 18$.

$\frac{\partial \phi_1}{\partial x} = 2x$ | $\frac{\partial \phi_2}{\partial x} = y - z$

$\frac{\partial \phi_1}{\partial y} = -2y$ | $\frac{\partial \phi_2}{\partial y} = x + z$

$\frac{\partial \phi_1}{\partial z} = -2z$ | $\frac{\partial \phi_2}{\partial z} = y - x$.

$\nabla \phi_1 = \vec{i} \frac{\partial \phi_1}{\partial x} + \vec{j} \frac{\partial \phi_1}{\partial y} + \vec{k} \frac{\partial \phi_1}{\partial z}$

$= 2x \vec{i} - 2y \vec{j} - 2z \vec{k}$

$(\nabla \phi_1)(6,4,3) = 12\vec{i} - 8\vec{j} - 6\vec{k}$.

$|\nabla \phi_1| = \sqrt{12^2 + (-8)^2 + (-6)^2}$

$= \sqrt{144 + 64 + 36}$

$= \sqrt{244}$.

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$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= (y-z)\vec{i} + (x+z)\vec{j} + (y-x)\vec{k}$$

$$(\nabla\phi_2)(6,4,3) = (4-3)\vec{i} + (6+3)\vec{j} + (4-6)\vec{k}$$

$$= \vec{i} + 9\vec{j} - 2\vec{k}$$

$$|\nabla\phi_2| = \sqrt{1^2 + 9^2 + (-2)^2}$$

$$= \sqrt{1+81+4} = \sqrt{86}.$$

The angle between the surfaces

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$= \frac{(12\vec{i} - 8\vec{j} - 6\vec{k})(\vec{i} + 9\vec{j} - 2\vec{k})}{\sqrt{244}\sqrt{86}}$$

$$= \frac{12 - 72 + 12}{\sqrt{244}\sqrt{86}} = \frac{48}{\sqrt{244}\sqrt{86}}.$$

$$\frac{48}{\sqrt{4\times61}\sqrt{86}} = \frac{48}{2\sqrt{61\times86}}$$

$$\theta = \cos^{-1}\left(\frac{48}{\sqrt{244}\sqrt{86}}\right)$$

$$= \cos^{-1}\left(\frac{24}{\sqrt{5246}}\right)$$

(3) Find the angle between the normals to the surface $xy = z^2$ at the points $(-2,-2,2)$ and $(1,9,-3)$

Given $\phi = xy - z^2$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$\frac{\partial\phi}{\partial x} = y$ | $\frac{\partial\phi}{\partial y} = x$ | $\frac{\partial\phi}{\partial z} = -2z$

$$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$(\nabla\phi)(-2,-2,2) = -2\vec{i} - 2\vec{j} - 4\vec{k} = \nabla\phi_1$$

$$(\nabla\phi)(1,9,-3) = 9\vec{i} + \vec{j} + 6\vec{k} = \nabla\phi_2$$

$$\cos\theta = \frac{\nabla\phi_1 \times \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{(-2\vec{i} - 2\vec{j} - 4\vec{k})(9\vec{i} + \vec{j} + 6\vec{k})}{\sqrt{(-2)^2 + (-2)^2 + (-4)^2}\sqrt{9^2 + 1^2 + 6^2}}$$

$$= \frac{-18 - 2 - 24}{\sqrt{4+4+16}\sqrt{81+1+36}}$$

$$= \frac{-44}{\sqrt{24}\sqrt{118}}$$

$$= \frac{-11}{\sqrt{177}}$$

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{77}}\right)$$

(4) Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1,1,1)$.

Given $\phi_1 = y^2 - x \log z - 1$

$\phi_2 = x^2 y - 2 + z$

$$\frac{\partial \phi_1}{\partial x} = -\log z$$

$$\frac{\partial \phi_1}{\partial y} = 2y$$

$$\frac{\partial \phi_1}{\partial z} = -x \cdot \frac{1}{z}$$

$$\frac{\partial \phi_2}{\partial x} = 2xy$$

$$\frac{\partial \phi_2}{\partial y} = x^2$$

$$\frac{\partial \phi_2}{\partial z} = 1.$$

$$(\nabla \phi_1) = \vec{i}\frac{\partial \phi_1}{\partial x} + \vec{j}\frac{\partial \phi_1}{\partial y} + \vec{k}\frac{\partial \phi_1}{\partial z}$$

$$= (\log z)\vec{i} + 2y\vec{j} - \frac{x}{z}\vec{k}$$

$$(\nabla\phi_1)(1,1,1) = 0\vec{i} + 2\vec{j} - \vec{k}$$

$$= 2\vec{j} - \vec{k}$$

lly

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= (2xy)\vec{i} + x^2\vec{j} + \vec{k}$$

$$(\nabla\phi_2)(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$$

$$|\nabla\phi_2| = \sqrt{4+1+1} = \sqrt{6}$$

$$|\nabla\phi_1| = \sqrt{4+1} = \sqrt{5}$$

$$\therefore \cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\vec{\nabla}\phi_1| \cdot |\nabla\phi_2|}$$

$$= \frac{(2\vec{j} - k)(2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{0+2-1}{\sqrt{30}} = \frac{1}{\sqrt{30}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$$

Orthogonal

If the 2 surfaces 'a' and 'b' are cut orthogonally then $\nabla\phi_1 \perp \nabla\phi_2$

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

Q1) Find 'a' and 'b' such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Given $\phi_1 = ax^2 - byz - (a+2)x$ — ①

$\phi_2 = 4x^2y + z^3 - 4$.

$\frac{\partial\phi_1}{\partial x} = 2ax - (a+2)$

$\frac{\partial\phi_1}{\partial y} = -bz$

$\frac{\partial\phi_1}{\partial z} = -by$

$\frac{\partial\phi_2}{\partial x} = 8xy$

$\frac{\partial\phi_2}{\partial y} = 4x^2$

$\frac{\partial\phi_2}{\partial z} = 3z^2$.

$\nabla\phi_1 = \vec{i}(2ax - a - 2) + \vec{j}(-bz) + \vec{k}(-by)$

$(\nabla\phi_1)(1,-1,2) = \vec{i}(2a - a - 2) + \vec{j}(-2b) + \vec{k}b$

$= (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$.

$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$

$(\nabla\phi_2)_{(1,-1,2)} = -8\vec{i}+4\vec{j}+12\vec{k}$

Given ϕ_1 and ϕ_2 are orthogonal

$\therefore \nabla\phi_1 \cdot \nabla\phi_2 = 0$

$[(a-2)\vec{i} - 2b\vec{j} + b\vec{k}]\,[-8\vec{i}+4\vec{j}+12\vec{k}] = 0$

$-8(a-2) - 8b + 12b = 0$

$-8a+16+4b = 0$

$\div$ by 4 $\quad -2a+4+b=0$

$2a-b-4=0$ —— ②

Since, the point $(1,-1,2)$ lies on the surface $\phi_1(x,y,z)=0$, we have

① $\Rightarrow$ $a(1)^2 - b(-1)(2) = (a+2)1$

$a+2b = a+2$

$a+2b-a-2=0$

$2b-2=0$

$b-1=0$

$\boxed{b=1}$ —— ③

Sub ③ in ①,

$2a-1-4=0$

$2a-5=0 \Rightarrow 2a=5$

$\boxed{a=5/2}$

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(2) Find the constants a and b, so that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ may cut orthogonally, at the point (1, −1, 2).

Soln

Two surfaces are said to cut orthogonally at a point of intersection, If the respective normals at the points are perpendicular (⊥).

Given, $\phi_1 = 5x^2 - 2yz - 9x = 0$ —— (1)

$\phi_2 = ax^2y + bz^3 - 4 = 0$ —— (2)

$\frac{\partial \phi_1}{\partial x} = 10x - 9$

$\frac{\partial \phi_1}{\partial y} = -2z$

$\frac{\partial \phi_1}{\partial z} = -2y.$

$\frac{\partial \phi_2}{\partial x} = 2axy$

$\frac{\partial \phi_2}{\partial y} = ax^2$

$\frac{\partial \phi_2}{\partial z} = 3bz^2$

$\nabla \phi_1 = \vec{i}\frac{\partial \phi_1}{\partial x} + \vec{j}\frac{\partial \phi_1}{\partial y} + \vec{k}\frac{\partial \phi_1}{\partial z}$

$= (10x - 9)\vec{i} - 2z\vec{j} - 2y\vec{k}$

$$(\nabla\phi_1)_{(1,-1,2)} = \vec{i} - 4\vec{j} + 2\vec{k}$$

$$\nabla\phi_2 = 2axy\,\vec{i} + ax^2\vec{j} + 3bz^2\vec{k}$$

$$(\nabla\phi_2)_{(1,-1,2)} = -2a\vec{i} + a\vec{j} + 12b\vec{k}$$

Since the surfaces cut orthogonally,

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$(\vec{i} - 4\vec{j} + 2\vec{k})(-2a\vec{i} + a\vec{j} + 12b\vec{k}) = 0$$

$$-2a - 4a + 24b = 0$$

$$-6a + 24b = 0$$

$\div$ by 6, $\quad -a + 4b = 0 \quad$ —— (3)

Since $(1,-1,2)$ is a point of intersection of the two surfaces, it lies on

$$ax^2y + bz^3 = 4$$

$$a \times 1 \times -1 + b \times 2^3 = 4$$

$$-a + 8b = 4 \quad \text{—— (4)}$$

(4) − (3)

$$-a + 8b = 4$$

$$a - 4b = 0$$

$$4b = 4$$

$$\boxed{b = 1}$$

③ ⇒ $-a+4\times1=0$

$-a=-4$

$\boxed{a=4}$

Scalar potential

① If $\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$, then find the value of ϕ.

Soln Given,

$$\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k} \quad \text{—①}$$

we know that

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} \quad \text{—②}$$

$$\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$$

Equating the co-efficient of $\vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = 2xyz \quad \text{—③}$$

$$\frac{\partial\phi}{\partial y} = x^2z \quad \text{—④}$$

$$\frac{\partial\phi}{\partial z} = x^2y \quad \text{—⑤}$$

Integrating ③ partially with respect to 'x', we get

$$\int \frac{\partial \phi}{\partial x} = \int 2xyz$$

$$\phi = 2yz \int x$$

$$\phi = 2yz \frac{x^2}{2} + C_1 \quad \text{—⑥}$$

$$= x^2yz + C_1$$

Integrating ④ partially with respect to 'y', we get

$$\int \frac{\partial \phi}{\partial y} = \int x^2 z$$

$$\phi = x^2yz + C_2 \quad \text{—⑦}$$

Integrating ⑤ partially with respect to 'z', we get

$$\int \frac{\partial \phi}{\partial z} = \int x^2 y$$

$$\phi = x^2yz + C_3 \quad \text{—⑧}$$

Combining ⑥, ⑦ and ⑧, we get

$$\phi = x^2yz + C,$$

where C is an arbitrary constants.

Q) If $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}$. find ϕ.

Soln Given

$$\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k} \quad \text{--- (1)}$$

we know that

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} \quad \text{--- (2)}$$

Equating (1) & (2)

$$\frac{\partial\phi}{\partial x} = yz \quad \text{.... (3)}$$

$$\frac{\partial\phi}{\partial y} = zx \quad \text{.... (4)}$$

$$\frac{\partial\phi}{\partial z} = xy \quad \text{.... (5)}$$

Integrating (3), with r. to 'x'

$$\int \frac{\partial\phi}{\partial x} = \int yz$$

$$\phi = xyz + C_1 \quad \text{... (6)}$$

Integrating (4), with r. to 'y'

$$\int \frac{\partial\phi}{\partial y} = \int zx$$

$$\phi = xyz + C_2 \quad \text{... (7)}$$

Integrating (5) with respect to 'z'

$$\int \frac{\partial \phi}{\partial z} = \int xy$$

$$\phi = xyz + c_3 \quad \ldots (8)$$

From (6), (7), (8)

$$\boxed{\phi = xyz + \text{constant.}}$$

(3) If $\vec{r}$ is the position vector of the point (x, y, z), $\vec{a}$ is a constant vector and $\phi = x^2 + y^2 + z^2$, prove that (i) grad $(\vec{r} \cdot \vec{a}) = \vec{a}$. (ii) $\vec{r} \cdot$ grad $\phi = 2\phi$.

Given $\phi = x^2 + y^2 + z^2$

$\vec{r}$ is the position vector of the point (x, y, z).

$\vec{a}$ is a constant vector.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$

$$\therefore \vec{r} \cdot \vec{a} = (x\vec{i} + y\vec{j} + z\vec{k})(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$$

$$\vec{r} \cdot \vec{a} = a_1x + a_2y + a_3z$$

we know

$$\nabla\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\therefore \text{grad}\,(\vec{r}\cdot\vec{a}) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$= \vec{a}.$$

[where

$$\frac{\partial}{\partial x}(\vec{r}\cdot\vec{a}) = a_1$$

$$\frac{\partial}{\partial y}(\vec{r}\cdot\vec{a}) = a_2$$

$$\frac{\partial}{\partial z}(\vec{r}\cdot\vec{a}) = a_3.$$

$$\therefore \boxed{\text{grad}\,(\vec{r}\cdot\vec{a}) = \vec{a}}$$

(ii) PT $\vec{r}\cdot\text{grad}\,\phi = 2\phi$.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\phi = x^2 + y^2 + z^2.$$

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = 2z$$

$$\therefore \text{grad}\,\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\vec{r}\cdot\text{grad}\,\phi = (x\vec{i} + y\vec{j} + z\vec{k})\cdot(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= 2x^2 + 2y^2 + 2z^2$$

$$= 2(x^2 + y^2 + z^2) = 2\phi.$$

Velocity and acceleration

Velocity $V = \frac{dr}{dt}$

Acceleration $a = \frac{d^2 r}{dt^2}$.

(1). If r is a vector of constant magnitude

P.T $r \cdot \frac{dr}{dt} = 0$.

Proof Let r be a vector of constant magnitude (modulus).

Then $|r| = r =$ constant

$r \cdot r = r^2 =$ constant.

Diff

$\frac{d}{dt}(r^2) = \frac{d}{dt}(\text{constant})$

$2r \cdot \frac{dr}{dt} = 0$.

$\therefore r \cdot \frac{dr}{dt} = 0$

but $r \neq 0$ $\therefore \frac{dr}{dt} = 0$.

$r \cdot \frac{dr}{dt} = 0 \Rightarrow \frac{dr}{dt}$ is $\perp$ to r.

(2) If r is a vector of constant direction

P.T $r \times \frac{dr}{dt} = 0$.

Proof:

Suppose 'r' is a vector of constant direction.

Let $r = rn$

where 'n' is a unit vector in the constant direction.

$$\frac{dr}{dt} = \frac{d}{dt}(rn) = \frac{dr}{dt} n$$

$$\therefore \quad r \times \frac{dr}{dt} = r \times \frac{dr}{dt} n$$

$r = rn$

$$= rn \times \frac{dr}{dt} n$$

$n \cdot n = 1$
$n \times n = 0$

$$= r \frac{dr}{dt} (n \times n)$$

$$r \times \frac{dr}{dt} = 0$$

(3). Find the velocity, speed, and the acceleration of the particle whose path is given by

(i) $r = 3\cos 2t \, \vec{i} + 2\sin 3t \, \vec{j}$.

(ii) $r = 4t\vec{i} - 4t\vec{j} + t\vec{k}$

Soln

(i) $r = 3\cos 2t\, i + 2\sin 3t\, k$

$$\frac{dr}{dt} = 3.(-\sin 2t).2i + 2(\cos 3t)3.k$$

$$\text{velocity}(V) = -6\sin 2t\, i + 6\cos 3t.k$$

$$\text{speed} = |v| = \sqrt{6^2\sin^2 2t + 6^2\cos^2 3t}$$

$$= \sqrt{36(\sin^2 2t + \cos^2 3t)}$$

Accelaration

$$a = \frac{d^2 r}{dt^2} = -6.(\cos 2t).2i + 6(-\sin 3t).3.k$$

$$a = -12\cos 2t\, i - 18\sin 3t\, k.$$

(ii) $r = 4t\, i - 4t\, j + t\, k.$

$$V = \frac{dr}{dt} = 4i - 4j + k$$

$$|V| = \sqrt{4^2 + 4^2 + 1} = \sqrt{16+16+1} = \sqrt{33}$$

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = 0$$

Q. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time. Find velocity and acceleration at $t=0$.

Soln

$$r = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{---} (1)$$

given $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$

$$(1) \Rightarrow r = e^{-t}\vec{i} + 2\cos 3t\,\vec{j} + 2\sin 3t\,\vec{k}$$

$$\frac{dr}{dt} = -e^{-t}\vec{i} + 2(-\sin 3t)\cdot 3\vec{j} + 2(\cos 3t)\cdot 3\vec{k}$$

$$V = \frac{dr}{dt} = -e^{-t}\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k} \quad \text{---} (1)$$

$$a = \frac{d^2r}{dt^2} = -e^{-t}\cdot(-1)\vec{i} - 6(\cos 3t)3\vec{j} + 6(-\sin 3t)\cdot 3\vec{k}$$

$$= e^{-t}\vec{i} - 18\cos 3t\,\vec{j} - 18\sin 3t\,\vec{k} \quad \text{---} (2)$$

(1) ⇒

$$\text{Velocity } (V \text{ at } t=0) = -e^{0}\vec{i} + 6(\sin 0)3\vec{j} + 6(\cos 0)\vec{k}$$

$\sin 0 = 0$
$\cos 0 = 1$

$$V_{(t=0)} = -\vec{i} + 6\vec{k}$$

Speed $= |V| = \sqrt{1+36} = \sqrt{37}$.

② ⇒ (Acceleration) $a_{t=0} = i - 18j$.

③ Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$.

Soln

$r = xi + yj + zk$

$\therefore r = 2\sin 3t\, i + 2\cos 3t\, j + 8t\, k$

$\frac{dr}{dt} = 2\cos 3t \cdot 3\, i + 2(-\sin 3t)\cdot 3\, j + 8k$

$\frac{dr}{dt} = V = 6\cos 3t\, i - 6\sin 3t\, j + 8k$.

$|V| = \sqrt{6^2\cos^2 3t + 6^2\sin^2 3t + 8^2}$

$= \sqrt{36(\cos^2 3t + \sin^2 3t) + 64}$

$= \sqrt{36+64} = \sqrt{100} = 10$

$a = \frac{d^2r}{dt^2} = 6(-\sin 3t)\cdot 3\, i - 6\cos 3t \cdot 3\, j$

$= -18(\sin 3t)\, i - 18\cos 3t\, j$

Divergence and curl — vector identities

Note

1) DIV $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

2) $\nabla \cdot \vec{F}$ is a scalar quantity.

3) curl $\vec{F} = \nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y}$
$+ \vec{k} \times \frac{\partial \vec{F}}{\partial z}$.

4) curl$\vec{F}$ is a vector point function.

5) If $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$, then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

6) If $\vec{F} = f(x)\vec{i} + g(y)\vec{j} + h(z)\vec{k}$, then

(i) $\nabla \cdot \vec{F} = \nabla (f(x)\vec{i} + g(y)\vec{j} + h(z)\vec{k})$

$= f'(x)\vec{i} + g'(y).\vec{j} + h(z).\vec{k}$.

(ii) $\nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$

$= \vec{i} \times f'(x)\, \vec{i} + \vec{j} \times f'(y)\, \vec{j} + \vec{k} \times f'(z) . \vec{k}$

$= f'(x)\, (\vec{i} \times \vec{i}) + f'(y)\, (\vec{j} \times \vec{j}) + f'(z) . (\vec{k} \times \vec{k})$

$\nabla \times \vec{F} = 0.$

Problems

① If $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$ then find (Div) $\nabla . \vec{F}$ and ② (curl) $\nabla \times \vec{F}$.

Given $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$

w.k.t

$\nabla . \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Here $F_1 = x^2,\ F_2 = y^2,\ F_3 = z^2$

$\therefore \nabla . \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$

$= 2x + 2y + 2z.$

$\nabla . \vec{F} = 2(x+y+z).$

② curl $\vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

---

3. Determine the constant 'a', If the divergence of the vector $\vec{F} = (x+z)\vec{i} + (3x+ay)\vec{j} + (x-5z)\vec{k}$ is zero.

Soln given $\nabla \cdot \vec{F} = 0$

$$\therefore \frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(3x+ay) + \frac{\partial}{\partial z}(x-5z) = 0.$$

$$1 + a - 5 = 0$$

$$a - 4 = 0 \Rightarrow \boxed{a=4}$$

(4) Find the divergence of the vector point function $xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$.

Soln.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2).$$

$$= y^2 + 2x^2z - 3y \times 2z$$

$$= y^2 + 2x^2z - 6yz.$$

---

(5) Find the curl of the vector point function $xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

Soln

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]$$

$$- j\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$= \vec{i}[-3z^2 - 2x^2y] - \vec{j}[0-0]$

$+ k[4xyz - 2xy]$

$= \vec{i}(-3z^2 - 2x^2y) + k(4xyz - 2xy).$

---

(6) If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find $div(curl\,\vec{F}) = 0$.

<u>soln</u> $curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$F_1 = x^3, \quad F_2 = y^3, \quad F_3 = z^3$

$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$

$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0]$

$= \vec{0}$

$div(curl\,\vec{F}) = div(\nabla \times \vec{F})$

$= \nabla(\nabla \times \vec{F})$

$= \nabla(0) = 0.$

Q. Calculate the curl of the vector (curl $\vec{F}$)

$\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$.

Soln. $\text{curl } \vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$= \vec{i}[0-y] - \vec{j}[z-0] + \vec{k}[0-x]$

$= -y\vec{i} - z\vec{j} - x\vec{k}$

$= -[y\vec{i} + z\vec{j} + x\vec{k}]$.

---

(7) Find $\nabla(\frac{1}{r}\vec{r})$.

Soln. we know that

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$\frac{1}{r}.\vec{r} = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$.

$\nabla\left[\frac{1}{r}\vec{r}\right] = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right)$

$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{1}{r} \times x\right)$$

$d(u \cdot v) = u \cdot dv + v \cdot du$

$$= \sum \left[\frac{1}{r} \cdot (1) + x \cdot \left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}\right]$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$= \sum \left[\frac{1}{r} - \frac{1}{r^2} \cdot x \cdot \frac{x}{r}\right]$$

$$= \sum \left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \sum \left[\frac{1}{r}\right] - \sum \left[\frac{x^2}{r^3}\right]$$

$$= \frac{1+1+1}{r} - \frac{x^2+y^2+z^2}{r^3}$$

$$= \frac{3}{r} - \frac{x^2+y^2+z^2}{r^3}$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \quad \left[r^2 = x^2+y^2+z^2\right]$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$\nabla\left(\frac{1}{r} \cdot \vec{r}\right) = \frac{2}{r}$$

Vector identities:

(1) $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \Sigma\left[\vec{i}\frac{\partial\phi}{\partial x}\right]$

(2) $\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{F}$

$= \vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{F}}{\partial z}$

$= \Sigma\left[\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right]$.

(3) $\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{F}$

$= \vec{i}\times\frac{\partial\vec{F}}{\partial x} + \vec{j}\times\frac{\partial\vec{F}}{\partial y} + \vec{k}\times\frac{\partial\vec{F}}{\partial z}$

$= \Sigma\left[\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right]$.

Problems:

① If ϕ is a scalar point function, then $\nabla\times(\nabla\phi) = \vec{0}$ (or)

Prove that $\text{Curl}(\text{grad}\,\phi) = \vec{0}$.

Soln

$\text{grad}\,\phi = \nabla\phi$

$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$\text{Curl}(\text{grad}\,\phi) = \nabla\times\nabla\phi$ ——— ①

w.k.t $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Scanned by CamScanner

$$\nabla \times \nabla \phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right] - \vec{j}\left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right] + \vec{k}\left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right]$$

$$= \vec{i}\,0 + \vec{j}\,0 + \vec{k}\,0$$

$$= \vec{0}.$$

---

② If $\vec{F}$ is a vector point function then $\nabla \cdot (\nabla \times \vec{F}) = 0$ (or) Prove that $div\,(curl\,\vec{F}) = 0$.

Soln

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$.

$$curl\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0.$$

Some standard results

If $\vec{u}, \vec{v}, \vec{w}$ are differentiable vector valued functions of the scalar variable t.

1) $\frac{d}{dt}(\vec{u}+\vec{v}) = \frac{d\vec{u}}{dt} + \frac{d\vec{v}}{dt}$

2) $\frac{d}{dt}(\vec{u}\cdot\vec{v}) = u\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\vec{v}$

3) $\frac{d}{dt}(\vec{u}\times\vec{v}) = u \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$

4) $\frac{d}{dt}(\phi\vec{u}) = \phi\frac{d\vec{u}}{dt} + \frac{d\phi}{dt}\vec{u}$

5) $\frac{d}{dt}(\vec{u}, \vec{v}, \vec{w}) = \frac{d\vec{u}}{dt}\cdot\vec{v}\times\vec{w} + \vec{u}\cdot\frac{d\vec{v}}{dt}\times\vec{w} + \vec{u}\cdot\vec{v}\times\frac{d\vec{w}}{dt}$.

6) $\frac{d}{dt}[\vec{u}\times(\vec{v}\times\vec{w})] = \frac{d\vec{u}}{dt}\times(\vec{v}\times\vec{w}) + \vec{u}\times\left(\frac{d\vec{v}}{dt}\times\vec{w}\right) + \vec{u}\times\left(\vec{v}\times\frac{d\vec{w}}{dt}\right)$

7) $\frac{d}{dt}\left(\frac{\vec{u}}{s}\right) = \frac{s\frac{d\vec{u}}{ds} - \vec{u}\frac{ds}{dt}}{s^2}$, where 's' is a scalar.

③ Prove div (u grad v) = $u \nabla^2 v$ + (grad u).(grad v)

<u>Soln</u> Given div (u grad v) = $u \cdot \nabla^2 v$ + (grad u)(grad v)

(i.e) $\nabla \cdot (u \nabla v) = u \cdot \nabla^2 v + \nabla u \cdot \nabla v$

Considler,

$$u \nabla v = u \left[\vec{i} \frac{\partial v}{\partial x} + \vec{j} \frac{\partial v}{\partial y} + \vec{k} \frac{\partial v}{\partial z} \right].$$

$$= \vec{i}\, u \frac{\partial v}{\partial x} + \vec{j}\, u \frac{\partial v}{\partial y} + \vec{k}\, u \frac{\partial v}{\partial z}$$

$$\nabla \cdot (u \nabla v) = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot$$

i.i=1
j.j=1
k.k=1

$$\left(\vec{i}\, u \frac{\partial v}{\partial x} + \vec{j}\, u \frac{\partial v}{\partial y} + \vec{k}\, u \cdot \frac{\partial v}{\partial z} \right)$$

$$= \frac{\partial}{\partial x}\left[u \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial y}\left[u \frac{\partial v}{\partial y} \right] + \frac{\partial}{\partial z}\left[u \frac{\partial v}{\partial z} \right]$$

$\frac{d}{dt}(uv) = u\frac{dv}{dt} + \frac{du}{dt}v$

$$= u \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + u \cdot \frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y}$$

$$+ u \cdot \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial z}.$$

$$= u \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}$$

$$+ \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z}$$

$$= u\nabla^2 v + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} \quad \ldots (1)$$

$$\nabla u = \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}$$

$$\nabla v = \vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}$$

$$\nabla u \cdot \nabla v = \left(\vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}\right)\left(\vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}\right)$$

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} \quad \ldots (2)$$

Substitute (2) in (1)

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u \cdot \nabla v$$

Hence proved

Problems based on solenoidal vector and irrotational vector:

Solenoidal Definition:

If $\vec{F}$ is a vector such that ($\operatorname{div}\vec{F} = 0$) $\boxed{\nabla\cdot\vec{F} = 0}$ at all points in a given region, then it is said to be solenoidal vector in that region.

Irrotational vector Definition

If $\vec{F}$ is a vector such that $\nabla \times \vec{F} = 0$ at all points in a given region, then it is said to be an irrotational vector in a region.

Problems:

① Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal

Soln Given: $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$.

To prove $\nabla \cdot \vec{F} = 0$.

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$
$$\left(z\vec{i} + x\vec{j} + y\vec{k}\right)$$
$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$
$$\nabla \cdot \vec{F} = 0.$$

$\therefore$ $\vec{F}$ is solenoidal.

---

② If $\vec{V} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Soln Given $\vec{V}$ is solenoidal

$\therefore \nabla \cdot \vec{V} = 0$.

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}(x+3y) + \vec{j}(y-2z) + \vec{k}(x+\lambda z)\right) = 0.$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\lambda z) = 0$$

$$1 + 1 + \lambda = 0$$

$$2 + \lambda = 0$$

$$\boxed{\lambda = -2}$$

) Find 'a' such that $(3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ is solenoidal.

Soln Given $\vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$.

Also, Given $\vec{F}$ is solenoidal

(ie) $\nabla \cdot \vec{F} = 0$.

$$\therefore \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\vec{i}(3x-2y+z) + \vec{j}(4x+ay-z) + \vec{k}(x-y+2z)\right)$$

$$= \frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z).$$

$3+a+2=0$

$5+a=0$

$\boxed{a=-5}$

---

(4) Determine $f(r)$, so that the vector $f(r)\vec{r}$ is solenoidal.

Soln

we know that

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$f(r)\,\vec{r} = f(r)\,x\vec{i} + f(r)\,y\vec{j} + f(r)\,z\vec{k}$$

Given : $f(r)\,\vec{r}$ is solenoidal.

$$\Rightarrow \nabla \cdot \left[f(r)\vec{r}\right] = 0$$

$$\Rightarrow \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(f(r)x\vec{i} + f(r)y\vec{j} + f(r).z\vec{k}\right) = 0.$$

$$\frac{\partial}{\partial x}\left[x f(r)\right] + \frac{\partial}{\partial y}\left[y f(r)\right] + \frac{\partial}{\partial z}\left[z f(r)\right] = 0$$

$$\Sigma \frac{\partial}{\partial x}\left[x f(r)\right] = 0$$

$$\Sigma \left[x.\left(f'(r).\frac{\partial r}{\partial x}\right) + f(r).(1)\right] = 0$$

$$\Sigma \left[x\left(f'(r).\frac{x}{r}\right) + f(r)\right] = 0$$

$$\Sigma \left[f'(r).\frac{x^2}{r} + f(r)\right] = 0$$

$$f'(r).\frac{x^2}{r} + f'(r).\frac{y^2}{r} + f'(r).\frac{z^2}{r} + f(r) + f(r) + f(r) = 0$$

(i.e.) $3f(r) + \frac{f'(r)}{r}\left[x^2+y^2+z^2\right] = 0$

$$3.f(r) + \frac{f'(r)}{r}.r^2 = 0 \qquad [x^2+y^2+z^2=r^2]$$

$$3f(r) + f'(r).r = 0$$

$$f'(r).r = -3f(r)$$

$$\frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating with respect to 'r' we get

$$\int \frac{f'(r)}{f(r)}dr = \int -\frac{3}{r}dr$$

$$\int \frac{f'(r)}{f(r)}dr = -3\int \frac{1}{r}dr$$

$$\log[f(r)] = -3\log r + \log c$$

$\log m + \log n = \log mn$

$$= \log(r^{-3}) + \log c$$

$$= \log\left(\frac{1}{r^3}\right) + \log c$$

$$\log f(r) = \log \frac{1}{r^3}.c$$

$$\therefore \boxed{f(r) = \frac{c}{r^3}}$$

5\. Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

Given : $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

To prove $\vec{F}$ is irrotational

(i.e) $\nabla \times \vec{F} = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \vec{i}[x - x] - \vec{j}[y - y] + k[z - z]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}.$$

Hence, $\vec{F}$ is irrotational.

6\. Find the constants a, b, c so that $\vec{F} = (x + ay + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

Soln: Given : $\vec{F}$ is irrotational.

(i.e) $\nabla \times \vec{f} = 0$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}[c+1] - \vec{j}[4-a] + \vec{k}[b-2] = 0\vec{i} - 0\vec{j} + 0\vec{k}$$

Equating both sides,

$c+1=0$ | $4-a=0$ | $b-2=0$

$c=-1$ | $a=4$ | $b=2$

---

⑦ If $\vec{A}$ is a constant vector, then prove that $\text{div}\, \vec{A} = 0$.

Given: $\vec{A}$ is a constant vector.

Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

d(constant) = 0

$$\frac{\partial A_1}{\partial x} = 0, \quad \frac{\partial A_2}{\partial y} = 0, \quad \frac{\partial A_3}{\partial z} = 0.$$

$$\nabla \cdot \vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(A_1\vec{i} + A_2\vec{j} + A_3\vec{k})$$

$$= \frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}(A_2) + \frac{\partial}{\partial z}(A_3)$$

$$= 0 + 0 + 0 = 0.$$

Hence $\text{div}\, \vec{A} = 0$.

8. If $\vec{A}$ is a constant vector, then

P.T curl $\vec{A} = \vec{0}$;

Soln. Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$.

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0]$$

$$= \vec{0}.$$

Hence curl $\vec{A} = \vec{0}$.

---

9. Prove that $\nabla^2(r^n) = n(n+1).r^{n-2}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$.

(Or)

P.T div(grad r^n) $= n(n+1) r^{n-2}$

Soln. Given: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$r = |\vec{r}| = |x\vec{i} + y\vec{j} + z\vec{k}|$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2 \quad \text{--- (1)}$$

Differentiating w.r.t. x, y, z, we get

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

|||ly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$\nabla(r) = \sum \frac{\partial}{\partial r}(\;)$

$$\nabla^2(r^n) = \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(r^n)\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n \cdot r^{n-1} \cdot \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n \cdot r^{n-1} \cdot \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n \cdot r^{n-1} \cdot x \cdot r^{-1}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n x r^{n-2}\right]$$

$$= \sum n \left\{\frac{\partial}{\partial x}\left[x r^{n-2}\right]\right\}$$

$$= \sum n \left\{x \cdot (n-2) r^{n-3} \cdot \frac{\partial r}{\partial x} + r^{n-2} \cdot (1)\right\}$$

$$= \sum n \left[x (n-2) r^{n-3} \frac{x}{r} + r^{n-2}\right]$$

$$= \Sigma n\left[x^2 (n-2) \cdot r^{n-3-1} + r^{n-2}\right]$$

$$= \Sigma n\left[x^2 (n-2) r^{n-4} + r^{n-2}\right]$$

$$= \Sigma\left[n(n-2) \cdot r^{n-4} \cdot x^2 + n \cdot r^{n-2}\right]$$

$\Sigma(r^2) = x^2+y^2+z^2$

$+n+n+n$

$$= n(n-2) \cdot r^{n-4} \cdot (x^2+y^2+z^2) + 3n \cdot r^{n-2}$$

$$= n(n-2) r^{n-4} \cdot r^2 + 3n r^{n-2}$$

$$= n(n-2) r^{n-2} + 3n r^{n-2}$$

$$= n r^{n-2} [n-2+3]$$

$$= n r^{n-2} [n+1]$$

$$\nabla^2(r^n) = n(n+1) \cdot r^{n-2}$$

Hence proved.

---

10. P.T $\nabla^2 f(r) = f''(r) + \left(\frac{2}{r}\right) \cdot f'(r)$.

Soln $\nabla^2 f(r) = \Sigma \frac{\partial^2}{\partial x^2} f(r)$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right)$$

$$= \Sigma \frac{\partial}{\partial x}\left(f'(r) \cdot \frac{\partial r}{\partial x}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(f'(r) \cdot \frac{x}{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\overset{u}{f'(r)} \cdot \overset{v}{x} \cdot \overset{w}{\frac{1}{r}}\right)$$

$$= \sum \Big[f'(r).x\left[-\frac{1}{r^2}\frac{\partial r}{\partial x}\right] + f'(r).(1).\frac{1}{r}$$

$$+ f''(r).\frac{\partial r}{\partial x} x \frac{1}{r}\Big]$$

$$= \sum \Big[- f'(r).x\frac{1}{r^2}.\frac{x}{r} + f'(r).\frac{1}{r}$$

$$+ f''(r).\frac{x}{r}.x\frac{1}{r}\Big]$$

$$= \sum \Big[- f'(r).\frac{1}{r^3}x^2 + f'(r).\frac{1}{r} + f''(r)\frac{1}{r^2}x^2\Big]$$

$$= - f'(r)\frac{1}{r^3}.(x^2+y^2+z^2) + \frac{3}{r}.f'(r)$$

$$+ f''(r).\frac{1}{r^2}(x^2+y^2+z^2).$$

$$= - f'(r)\frac{1}{r^3}.(r^2) + \frac{3}{r}f'(r) + f''(r)\frac{1}{r^2}r^2$$

$$= - f'(r).\frac{1}{r} + \frac{3}{r}f'(r) + f''(r)$$

$$= f''(r) + \frac{2}{r}f'(r).$$

---

(11). P.T $\nabla \cdot \nabla\phi = \nabla^2\phi$

$$\nabla \cdot \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \nabla^2 \phi$$

12 Find $\nabla^2(r^2)$

Soln $\nabla^2(r^2) = \sum \frac{\partial^2}{\partial x^2}(r^2)$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial (r^2)}{\partial x}\right)$$

$$= \sum \frac{\partial}{\partial x}\left[2r \cdot \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[2r \cdot \frac{x}{r}\right]$$

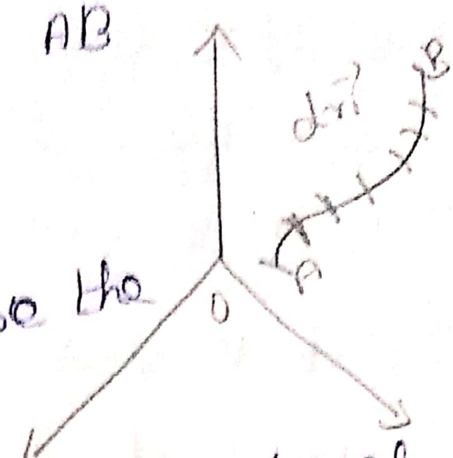
$$= \sum \frac{\partial}{\partial x}[2x]$$

$$= \sum(2) = 2+2+2 = 6$$

Line Integral

Let $\vec{F}$ be a vector field in space and Let AB be a curve described in the sense A to B.

Divide the curve AB into n elements $\vec{dr_1}, \vec{dr_2}, \ldots \vec{dr_n}$.



Let $\vec{F_1}, \vec{F_2}, \ldots \vec{F_n}$ be the values of this vector at the junction points of the vectors $\vec{dr_1}, \vec{dr_2}, \ldots \vec{dr_n}$.

Then, the sum $\lim_{n \to \infty} \sum_A^B \vec{F_n} \cdot \vec{dr_n} = \int_A^B \vec{F} \cdot \vec{dr}$ is called the line integral.

If the line Integral is along the curve C then it is denoted by $\int_C \vec{F} \cdot \vec{dr}$.

where C is a closed curve.

Note: 1

The scalar function ϕ is called the scalar potential of the vector field $\vec{F}$.

2) If C is a closed curve, then $\phi(A) = \phi(B)$. since A, B coinside. In this case $\int_C \nabla\phi \cdot d\vec{r} = 0$.

3). $\vec{F}$ is conservative, then curl $\vec{F}$ = curl grad $\phi = \vec{0}$.

4) If a vector point function $\vec{F}$ is conservative, then there exists a scalar point function ϕ such that $\vec{F} = \nabla\phi$.

Problems.

① If $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x = t, y = t^2, z = t^3$.

Soln The end points are $(0,0,0)$ and $(1,1,1)$.

These points correspond to $t = 0$ and $t = 1$

$x = t$	$y = t^2$	$z = t^3$
$dx = dt$	$dy = 2t\,dt$	$dz = 3t^2 dt$.

$\int \vec{F} \cdot d\vec{r} = \int (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$

$$= \int_0^1 (3t^2+6t^2)\,dt - 14\, t^2 \cdot t^3 (2t\,dt) + 20 \cdot t (t^3)^2 (3t^2\,dt)$$

$$= \int_0^1 9t^2\,dt - 14\, t^5 \cdot (2t\,dt) + 20\, t \cdot t^6 \cdot (3t^2\,dt)$$

$$= \int_0^1 (9t^2 - 28\, t^6 + 60\, t^9)\,dt$$

$$= \left[9 \cdot \frac{t^3}{3} - 28 \cdot \frac{t^7}{7} + 60 \cdot \frac{t^{10}}{10} \right]_0^1$$

$$= \left[3t^3 - 4t^7 + 6t^{10} \right]_0^1$$

$$= (3 - 4 + 6) - 0 = 5$$

---

2. If $\vec{F} = x^2 \vec{i} + y^3 \vec{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ in the xy plane along the curve 'c' $y = x^2$ from the point (0,0) to (1,1)

The end points are (0,0) & (1,1). These points corresponds to $t=0$ and $t=1$.

given $y = x^2 \Rightarrow dy = 2x\,dx$.

$$\int \vec{F} \cdot d\vec{r} = \int_C x^2\,dx + y^3\,dy$$

$$= \int_C x^2 dx + (x^2)^3 \cdot 2x\,dx$$

$$= \int_0^1 (x^2 + x^6 \cdot 2x)\,dx$$

$$= \int_0^1 (x^2 + 2 \cdot x^7)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^8}{8_4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$$

---

3) State the physical interpretation of the integral $\int_A^B \vec{F} \cdot d\vec{r}$.

Soln physically $\int_A^B \vec{F} \cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, in displacing a particle from A to B along the curve C.

Triple Integrals

1. (a) *If $\mathcal{U}$ is any solid (in space), what does the triple integral* $\iiint_{\mathcal{U}} 1\ dV$ *represent? Why?*

   Solution. Remember that we are thinking of the triple integral $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ as a limit of Riemann sums, obtained from the following process:

   1. Slice the solid $\mathcal{U}$ into small pieces.
   2. In each piece, the value of f will be approximately constant, so multiply the value of f at any point by the volume ΔV of the piece. (It's okay to approximate the volume ΔV.)
   3. Add up all of these products. (This is a Riemann sum.)
   4. Take the limit of the Riemann sums as the volume of the pieces tends to 0.

   Now, if f is just the function $f(x,y,z) = 1$, then in Step 2, we end up simply multiplying 1 by the volume of the piece, which gives us the volume of the piece. So, in Step 3, when we add all of these products up, we are just adding up the volume of all the small pieces, which gives the volume of the whole solid.

   So, $\iiint_{\mathcal{U}} 1\ dV$ represents the $\boxed{\text{volume of the solid } \mathcal{U}}$.

   (b) *Suppose the shape of a solid object is described by the solid $\mathcal{U}$, and $f(x,y,z)$ gives the density of the object at the point (x,y,z) in kilograms per cubic meter. What does the triple integral* $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ *represent? Why?*

   Solution. Following the process described in (a), in Step 2, we multiply the approximate density of each piece by the volume of that piece, which gives the approximate mass of that piece. Adding those up gives the approximate mass of the entire solid object, and taking the limit gives us the exact $\boxed{\text{mass of the solid object}}$.

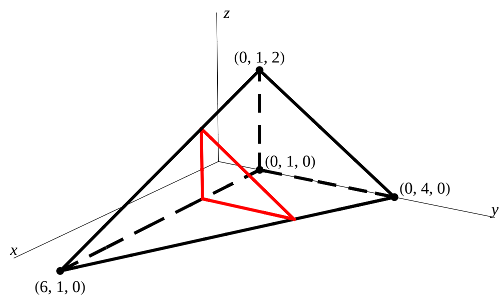
2. *Let $\mathcal{U}$ be the solid tetrahedron bounded by the planes $x = 0$, $y = 1$, $z = 0$, and $x + 2y + 3z = 8$. (The vertices of this tetrahedron are $(0,1,0)$, $(0,1,2)$, $(6,1,0)$, and $(0,4,0)$). Write the triple integral* $\iiint_{\mathcal{U}} f(x,y,z)\ dV$ *as an iterated integral.*

   Solution. We'll do this in all 6 possible orders. Let's do it by writing the outer integral first, which means we think of slicing. There are three possible ways to slice: parallel to the yz-plane, parallel to the xz-plane, and parallel to the xy-plane.

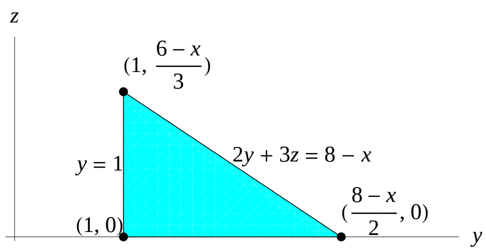
   (a) **Slice parallel to the yz-plane.**

   If we do this, we are slicing the interval $[0,6]$ on the x-axis, so the outer (single) integral will be $\int_0^6$ something dx.

   To write the inner two integrals, we look at a typical slice and describe it. When we do this, we think of x as being constant (since, within a slice, x is constant). Here is a typical slice:



Each slice is a triangle, with one edge on the plane $y = 1$, one edge on the plane $z = 0$, and one edge on the plane $x + 2y + 3z = 8$. (Since we are thinking of x as being constant, we might rewrite this last equation as $2y + 3z = 8 - x$.) Here's another picture of the slice, in 2D:



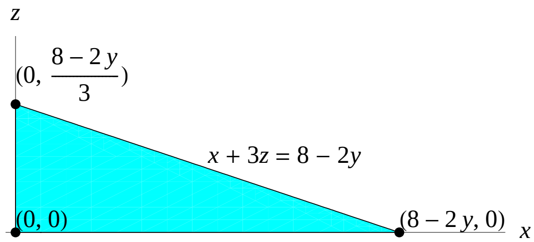
Now, we write a (double) iterated integral that describes this region. This will make up the inner two integrals of our final answer. There are two ways to do this:

i. If we slice vertically, we are slicing the interval $\left[1, \frac{8-x}{2}\right]$ on the y-axis, so the outer integral (of the two we are working on) will be $\displaystyle\int_1^{(8-x)/2}$ something dy. Each slice goes from $z = 0$ to the line $2y + 3z = 8 - x$ (since we're trying to describe z within a vertical slice, we'll rewrite this as $z = \frac{8-x-2y}{3}$), so the inner integral will be $\displaystyle\int_0^{(8-x-2y)/3} f(x, y, z)\ dz$. This gives us the iterated integral $\boxed{\displaystyle\int_0^6 \int_1^{(8-x)/2} \int_0^{(8-x-2y)/3} f(x, y, z)\ dz\ dy\ dx}$.

ii. If we slice horizontally, we are slicing the interval $\left[0, \frac{6-x}{3}\right]$ on the z-axis, so the outer integral (of the two we are working on) will be $\displaystyle\int_0^{(6-x)/3}$ something dz. Each slice goes from $y = 1$ to the line $2y + 3z = 8 - x$ (since we are trying to describe y in a horizontal slice, we'll rewrite this as $y = \frac{8-x-3z}{2}$), so the inner integral will be $\displaystyle\int_1^{(8-x-3z)/2} f(x, y, z)\ dy$. This gives the final answer $\boxed{\displaystyle\int_0^6 \int_0^{(6-x)/3} \int_1^{(8-x-3z)/2} f(x, y, z)\ dy\ dz\ dx}$.

(b) **Slice parallel to the xz-plane.**

If we do this, we are slicing the interval $[1, 4]$ on the y-axis. So, our outer (single) integral will be $\displaystyle\int_1^4$ something dy. Each slice is a triangle with edges on the planes $x = 0$, $z = 0$, and $x+2y+3z = 8$ (or $x + 3z = 8 - 2y$). Within a slice, y is constant, so we can just look at x and z:



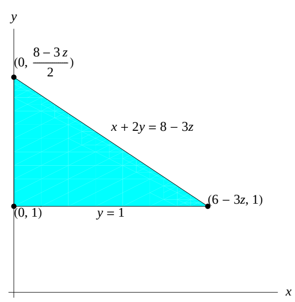
Our inner two integrals will describe this region.

i. If we slice vertically, we are slicing the interval $[0, 8-2y]$ on the x-axis, so the outer integral (of the two we're working on) will be $\int_0^{8-2y}$ something dx. Each slice goes from $z = 0$ to $z = \frac{8-2y-x}{3}$, which gives the iterated integral $\boxed{\int_1^4 \int_0^{8-2y} \int_0^{(8-2y-x)/3} f(x, y, z)\ dz\ dx\ dy}$.

ii. If we slice horizontally, we are slicing the interval $\left[0, \frac{8-2y}{3}\right]$ on the z-axis, so the outer integral (of the two we're working on) will be $\int_0^{(8-2y)/3}$ something dz. Each slice goes from $x = 0$ to $x = 8-2y-3z$, which gives the iterated integral $\boxed{\int_1^4 \int_0^{(8-2y)/3} \int_0^{8-2y-3z} f(x, y, z)\ dx\ dz\ dy}$.

(c) **Slice parallel to the xy-plane.**

If we do this, we are slicing the interval $[0, 2]$ on the z-axis, so the outer (single) integral will be $\int_0^2$ something dz. Each slice is a triangle with edges on the planes $x = 0$, $y = 1$, and $x+2y+3z = 8$ (or $x + 2y = 8 - 3z$). Within a slice, z is constant, so we can just look at x and y:

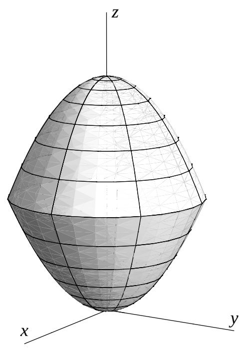


Our inner two integrals will describe this region.

i. If we slice vertically, we are slicing the interval $[0, 6-3z]$ on the x-axis, so the outer integral (of the two we're working on) will be $\int_0^{6-3z}$ something dx. Each slice will go from $y = 1$ to the line $x + 2y = 8 - 3z$ (which we write as $y = \frac{8-3z-x}{2}$ since we're trying to describe y), which gives us the final integral $\boxed{\int_0^2 \int_0^{6-3z} \int_1^{(8-3z-x)/2} f(x, y, z)\ dy\ dx\ dz}$.

ii. If we slice horizontally, we are slicing the interval $\left[1, \frac{8-3z}{2}\right]$ on the y-axis, so the outer integral (of the two we're working on) will be $\int_1^{(8-3z)/2}$ something dy. Each slice will go from $x = 0$ to $x + 2y = 8 - 3z$ (which we write as $x = 8 - 3z - 2y$ since we're trying to describe x), which gives us the answer $\boxed{\int_0^2 \int_1^{(8-3z)/2} \int_0^{8-3z-2y} f(x, y, z)\ dx\ dy\ dz}$.

3. *Let $\mathcal{U}$ be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$. (Note: The paraboloids intersect where $z = 4$.) Write $\iiint_{\mathcal{U}} f(x, y, z)\ dV$ as an iterated integral in the order $dz\ dy\ dx$.*

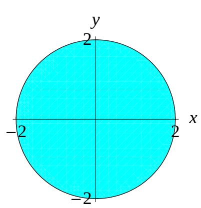


Solution. We can either do this by writing the inner integral first or by writing the outer integral first. In this case, it's probably easier to write the inner integral first, but we'll show both methods.

- **Writing the inner integral first:**

  We are asked to have our inner integral be with respect to z, so we want to describe how z varies along a vertical line (where x and y are fixed) to write the inner integral. We can see that, along any vertical line through the solid, z goes from the bottom paraboloid ($z = x^2 + y^2$) to the top paraboloid ($z = 8 - (x^2 + y^2)$), so the inner integral will be $\int_{x^2+y^2}^{8-(x^2+y^2)} f(x, y, z)\ dz$.

  To write the outer two integrals, we want to describe the projection of the region onto the xy-plane. (A good way to think about this is, if we moved our vertical line around to go through the whole solid, what x and y would we hit? Alternatively, if we could stand at the "top" of the z-axis and look "down" at the solid, what region would we see?) In this case, the widest part of the solid is where the two paraboloids intersect, which is $z = 4$ and $x^2 + y^2 = 4$. Therefore, the projection is the region $x^2 + y^2 \leq 4$, a disk in the xy-plane:



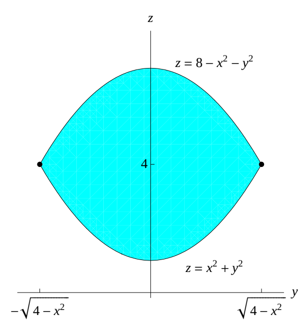
  We want to write an iterated integral in the order $dy\ dx$ to describe this region, which means we should slice vertically. We slice $[-2, 2]$ on the x-axis, so the outer integral will be $\int_{-2}^{2}$ something dx.

Along each slice, y goes from the bottom of the circle ($y = -\sqrt{4-x^2}$) to the top ($y = \sqrt{4-x^2}$), so we get the iterated integral $\boxed{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-(x^2+y^2)} f(x,y,z)\ dz\ dy\ dx}$.

- **Writing the outer integral first:**

  We are asked to have our outer integral be with respect to x, so we want to make slices parallel to the yz-plane. This amounts to slicing the interval $[-2, 2]$ on the x-axis, so the outer integral will be $\int_{-2}^{2}$ something dx.

  Each slice is a region bounded below by $z = x^2 + y^2$ and above by $z = (8 - x^2) - y^2$. (Remember that, within a slice, x is constant.) Note that these curves intersect where $x^2 + y^2 = (8 - x^2) - y^2$, or $2y^2 = 8 - 2x^2$. This happens at $y = \pm\sqrt{4-x^2}$. At either of these y-values, $z = 4$. So, here is a picture of the region:

  The two integrals describing this region are supposed to be in the order $dz\ dy$, which means we are slicing vertically. Slicing vertically amounts to slicing the interval $[-\sqrt{4-x^2}, \sqrt{4-x^2}]$ on the y-axis, so the outer integral (of these two integrals) will be $\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$ something dy. Along each vertical slice, z goes from $x^2 + y^2$ to $8 - (x^2 + y^2)$, so we get the final iterated integral $\boxed{\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-(x^2+y^2)} f(x,y,z)\ dz\ dy\ dx}$.

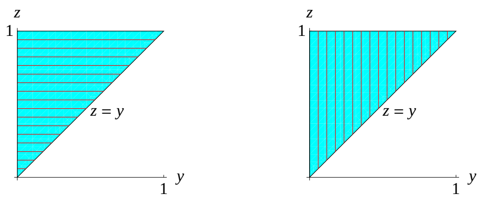
4. In this problem, we'll look at the iterated integral $\int_0^1 \int_0^z \int_{y^2}^1 f(x,y,z)\ dx\ dy\ dz$.

   (a) *Rewrite the iterated integral in the order $dx\ dz\ dy$.*

   Solution. One strategy is to draw the solid region of integration and then write the integral as we did in #3. However, drawing the solid region of integration is rather challenging, so here's another approach.

   Remember that we can think of a triple integral as either a single integral of a double integral or a double integral of a single integral, and we know how to change the order of integration in a double integral. (See, for instance, #5 on the worksheet "Double Integrals over General Regions".) This effectively means that we can change the order of the inner two integrals by thinking of them together as a double integral, or we can change the order of the outer two integrals by thinking of them together as a double integral.

For this question, we just need to change the order of the outer two integrals, so we focus on those. They are $\int_0^1 \int_0^z$ stuff $dy\ dz$.(1) Since this integral is $dy\ dz$, we should visualize the yz-plane. The fact that the outer integral is $\int_0^1$ something dz tells us that we are slicing the interval $[0,1]$ on the z-axis. The fact that the inner integral is $\int_0^z$ stuff dy tells us that each slice starts at $y=0$ and goes to $y=z$. So, our region (with horizontal slices) looks like the picture on the left:



To change the order of integration, we want to use vertical slices (the picture on the right). Now, we are slicing the interval $[0,1]$ on the y-axis, so the outer integral will be $\int_0^1$ something dy. Each slice has its bottom edge on $z=y$ and its top edge on $z=1$, so we rewrite $\int_0^1 \int_0^z$ stuff $dy\ dz$ as $\int_0^1 \int_y^1$ stuff $dz\ dy$. Remembering that "stuff" was the inner integral $\int_{y^2}^1 f(x,y,z)\ dx$, this gives us the iterated integral $\boxed{\int_0^1 \int_y^1 \int_{y^2}^1 f(x,y,z)\ dx\ dz\ dy}$.

(b) *Rewrite the iterated integral in the order $dz\ dy\ dx$.*

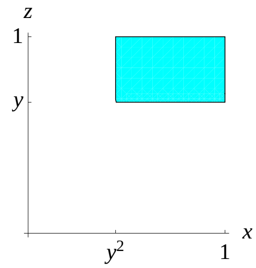
Solution. Let's continue from (a). As explained there, we can change the order of the outer two integrals or of the inner two integrals. From (a), we have our iterated integral in the order $dx\ dz\ dy$. If we change the order of the inner two integrals, then we'll have our iterated integral in the order $dz\ dx\ dy$. If we then change the order of the outer two integrals of this, we'll get it into the order $dz\ dy\ dx$. So, we really have two steps.

- **Step 1: Change the order of the inner double integral from (a).**

  We had $\int_0^1 \int_y^1 \int_{y^2}^1 f(x,y,z)\ dx\ dz\ dy$, so we are going to focus on the inner double integral $\int_y^1 \int_{y^2}^1 f(x,y,z)\ dx\ dz$. Remember that, since this is the inner double integral and y is the outer variable, we now think of y as being a constant.(2) Then, the region of integration of the integral $\int_y^1 \int_{y^2}^1 1 f(x,y,z)\ dz\ dz$ is just a rectangle (sliced horizontally):

(1)Here, "stuff" is the inner integral $\int_{y^2}^1 f(x,y,z)\ dx$.

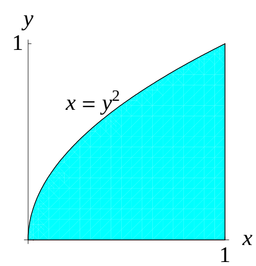
(2)In fact, we should think of y as being a constant between 0 and 1, since the outer integral has y going from 0 to 1.



If we change to slicing horizontally, we rewrite this as $\int_{y^2}^{1}\int_{y}^{1} f(x,y,z)\; dz\; dx$.[(3)] Putting the outer integral back, we get the iterated integral $\int_0^1 \int_{y^2}^{1}\int_{y}^{1} f(x,y,z)\; dz\; dx\; dy$.

- **Step 2: Change the order of the outer double integral.**

  Now, we're working with $\int_0^1 \int_{y^2}^{1}\int_{y}^{1} f(x,y,z)\; dz\; dx\; dy$, and we want to change the order of the outer double integral, which is $\int_0^1 \int_{y^2}^{1}$ stuff $dx\; dy$ with "stuff" being the inner integral $\int_y^1 f(x,y,z)\; dz$. The region of integration of $\int_0^1 \int_{y^2}^{1}$ stuff $dx\; dy$ looks like this (with horizontal slices):

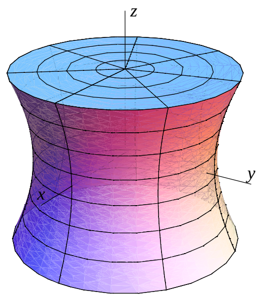
  

  If we change to slicing vertically, then the integral becomes $\int_0^1 \int_0^{\sqrt{x}}$ stuff $dy\; dx$. Remembering that "stuff" was the inner integral, we get our final answer $\boxed{\int_0^1 \int_0^{\sqrt{x}} \int_y^1 f(x,y,z)\; dz\; dy\; dx}$.

5. *Let $\mathcal{U}$ be the solid contained in $x^2 + y^2 - z^2 = 16$ and lying between the planes $z = -3$ and $z = 3$. Sketch $\mathcal{U}$ and write an iterated integral which expresses its volume. In which orders of integration can you write just a single iterated integral (as opposed to a sum of iterated integrals)?*

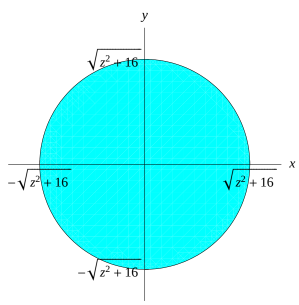
   Solution. Here is a picture of $\mathcal{U}$:[(4)]

(3) Another way of thinking about it is that we're using Fubini's Theorem.
(4) To remember how to sketch things like this, look back at #3 of the worksheet "Quadric Surfaces".



By #1(a), we know that a triple integral expressing the volume of $\mathcal{U}$ is $\iiint_{\mathcal{U}} 1\ dV$. We are asked to rewrite this as an iterated integral. Let's think about slicing the solid (which means we'll write the outer integral first). If we slice parallel to the xy-plane, then we are really slicing $[-3,3]$ on the z-axis, and the outer integral is $\int_{-3}^{3}$ something dz.

We use our inner two integrals to describe a typical slice. Each slice is just the disk enclosed by the circle $x^2+y^2=z^2+16$, which is a circle of radius $\sqrt{z^2+16}$:

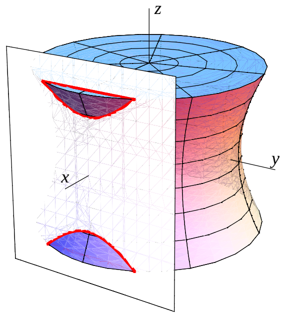


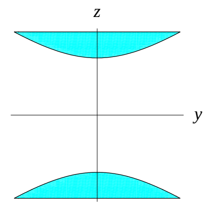
We can slice this region vertically or horizontally; let's do it vertically. This amounts to slicing $\left[-\sqrt{z^2+16}, \sqrt{z^2+16}\right]$ on the x-axis, so the outer integral is $\int_{-\sqrt{z^2+16}}^{\sqrt{z^2+16}}$ something dx. Along each slice, y goes from the bottom of the circle ($y=-\sqrt{z^2+16-x^2}$) to the top of the circle ($y=\sqrt{z^2+16-x^2}$). So, the inner integral is $\int_{-\sqrt{z^2+16-x^2}}^{\sqrt{z^2+16-x^2}} f(x,y,z)\ dy$.

Putting this all together, we get the iterated integral $\boxed{\int_{-3}^{3}\int_{-\sqrt{z^2+16}}^{\sqrt{z^2+16}}\int_{-\sqrt{z^2+16-x^2}}^{\sqrt{z^2+16-x^2}} 1\ dy\ dx\ dz}$.

We are also asked in which orders we can write just a single iterated integral. Clearly, we've done so with the order $dy\ dx\ dz$. We also could have with $dx\ dy\ dz$, since that would just be slicing the same disk horizontally.

If we had dx or dy as our outer integral, then we would need to write multiple integrals. For instance, if we slice the hyperboloid parallel to the yz-plane, some slices would look like this:





We would need to write a sum of integrals to describe such a slice. So, we can write a single iterated integral only in the orders $\boxed{dy\ dx\ dz \text{ and } dx\ dy\ dz}$.

Double integrals

Notice: this material must not be used as a substitute for attending the lectures

0.1 What is a double integral?

Recall that a **single integral** is something of the form

$$\int_a^b f(x)\,dx$$

A **double integral** is something of the form

$$\iint_R f(x,y)\,dx\,dy$$

where R is called the **region of integration** and is a region in the (x, y) plane. The double integral gives us the volume under the surface $z = f(x, y)$, just as a single integral gives the area under a curve.

0.2 Evaluation of double integrals

To evaluate a double integral we do it in stages, starting from the inside and working out, using our knowledge of the methods for single integrals. The easiest kind of region R to work with is a rectangle. To evaluate

$$\iint_R f(x,y)\,dx\,dy$$

proceed as follows:

- work out the limits of integration if they are not already known
- work out the inner integral for a typical y
- work out the outer integral

0.3 Example

Evaluate

$$\int_{y=1}^{2}\int_{x=0}^{3}(1+8xy)\,dx\,dy$$

Solution. In this example the "inner integral" is $\int_{x=0}^{3}(1+8xy)\,dx$ with y treated as a constant.

$$\begin{aligned}
\text{integral} &= \int_{y=1}^{2}\left(\underbrace{\int_{x=0}^{3}(1+8xy)\,dx}_{\text{work out treating } y \text{ as constant}}\right)dy \\
&= \int_{y=1}^{2}\left[x+\frac{8x^2y}{2}\right]_{x=0}^{3}dy \\
&= \int_{y=1}^{2}(3+36y)\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \left[3y + \frac{36y^2}{2}\right]_{y=1}^{2} \\
&= (6+72) - (3+18) \\
&= 57
\end{aligned}
$$

0.4 Example

Evaluate

$$\int_0^{\pi/2}\int_0^1 y \sin x \, dy \, dx$$

Solution.

$$
\begin{aligned}
\text{integral} &= \int_0^{\pi/2}\Big(\int_0^1 y \sin x \, dy\Big)\, dx \\
&= \int_0^{\pi/2} \left[\frac{y^2}{2}\sin x\right]_{y=0}^{1} dx \\
&= \int_0^{\pi/2} \tfrac{1}{2}\sin x \, dx \\
&= \left[-\tfrac{1}{2}\cos x\right]_{x=0}^{\pi/2} = \tfrac{1}{2}
\end{aligned}
$$

0.5 Example

Find the volume of the solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x, y) : \ 0 \le x \le 1 \ \ 0 \le y \le 2\}$.
Solution. The volume under any surface $z = f(x, y)$ and above a region R is given by

$$V = \iint_R f(x, y)\, dx\, dy$$

In our case

$$
\begin{aligned}
V &= \int_0^2\int_0^1 (4 - x - y)\, dx\, dy \\
&= \int_0^2 \left[4x - \tfrac{1}{2}x^2 - yx\right]_{x=0}^{1} dy = \int_0^2 (4 - \tfrac{1}{2} - y)\, dy \\
&= \left[\frac{7y}{2} - \frac{y^2}{2}\right]_{y=0}^{2} = (7-2) - (0) = 5
\end{aligned}
$$

The double integrals in the above examples are the easiest types to evaluate because they are examples in which all four limits of integration are constants. This happens when the region of integration is rectangular in shape. In non-rectangular regions of integration the limits are not all constant so we have to get used to dealing with non-constant limits. We do this in the next few examples.

0.6 Example

Evaluate

$$\int_0^2 \int_{x^2}^{x} y^2 x \, dy \, dx$$

Solution.

$$\begin{aligned} \text{integral} &= \int_0^2 \int_{x^2}^{x} y^2 x \, dy \, dx \\ &= \int_0^2 \left[\frac{y^3 x}{3} \right]_{y=x^2}^{y=x} dx \\ &= \int_0^2 \left(\frac{x^4}{3} - \frac{x^7}{3} \right) dx = \left[\frac{x^5}{15} - \frac{x^8}{24} \right]_0^2 \\ &= \frac{32}{15} - \frac{256}{24} = -\frac{128}{15} \end{aligned}$$

0.7 Example

Evaluate

$$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos \frac{y}{x} \, dy \, dx$$

Solution. Recall from elementary calculus the integral $\int \cos my \, dy = \frac{1}{m} \sin my$ for m independent of y. Using this result,

$$\begin{aligned} \text{integral} &= \int_{\pi/2}^{\pi} \left[\frac{1}{x} \frac{\sin \frac{y}{x}}{\frac{1}{x}} \right]_{y=0}^{y=x^2} dx \\ &= \int_{\pi/2}^{\pi} \sin x \, dx = [-\cos x]_{x=\pi/2}^{\pi} = 1 \end{aligned}$$

0.8 Example

Evaluate

$$\int_1^4 \int_0^{\sqrt{y}} e^{x/\sqrt{y}} \, dx \, dy$$

Solution.

$$\begin{aligned} \text{integral} &= \int_1^4 \left[\frac{e^{x/\sqrt{y}}}{1/\sqrt{y}} \right]_{x=0}^{x=\sqrt{y}} dy \\ &= \int_1^4 (\sqrt{y} e - \sqrt{y}) \, dy = (e-1) \int_1^4 y^{1/2} \, dy \\ &= (e-1) \left[\frac{y^{3/2}}{3/2} \right]_{y=1}^{4} = \frac{2}{3}(e-1)(8-1) \\ &= \frac{14}{3}(e-1) \end{aligned}$$

0.9 Evaluating the limits of integration

When evaluating double integrals it is very common **not** to be told the limits of integration but simply told that the integral is to be taken over a certain specified region R in the (x, y) plane. In this case you need to work out the limits of integration for yourself. Great care has to be taken in carrying out this task. The integration can in principle be done in two ways: (i) integrating first with respect to x and then with respect to y, or (ii) first with respect to y and then with respect to x. The limits of integration in the two approaches will in general be quite different, but both approaches must yield the same answer. Sometimes one way round is considerably harder than the other, and in some integrals one way works fine while the other leads to an integral that cannot be evaluated using the simple methods you have been taught. There are no simple rules for deciding which order to do the integration in.

0.10 Example

Evaluate

$$\iint_D (3 - x - y)\, dA \qquad [dA \text{ means } dxdy \text{ or } dydx]$$

where D is the triangle in the (x, y) plane bounded by the x-axis and the lines $y = x$ and $x = 1$.

Solution. A good diagram is essential.

Method 1 : do the integration with respect to x first. In this approach we select a typical y value which is (for the moment) considered fixed, and we draw a **horizontal** line across the region D; this horizontal line intersects the y axis at the typical y value. Find out the values of x (they will depend on y) where the horizontal line **enters** and **leaves** the region D (in this problem it enters at $x = y$ and leaves at $x = 1$). These values of x will be the limits of integration for the inner integral. Then you determine what values y has to range between so that the horizontal line sweeps the entire region D (in this case y has to go from 0 to 1). This determines the limits of integration for the outer integral, the integral with respect to y. For this particular problem the integral becomes

$$\begin{aligned}
\iint_D (3 - x - y)\, dA &= \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy \\
&= \int_0^1 \left[3x - \frac{x^2}{2} - yx \right]_{x=y}^{x=1} dy \\
&= \int_0^1 \left(\left(3 - \tfrac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right) \right) dy \\
&= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5y}{2} - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} \\
&= \frac{5}{2} - 2 + \frac{1}{2} = 1
\end{aligned}$$

Method 2 : do the integration with respect to y first and then x. In this approach we select a "typical x" and draw a vertical line across the region D at that value of x.

Vertical line enters D at $y = 0$ and leaves at $y = x$. We then need to let x go from 0 to 1 so that the vertical line sweeps the entire region. The integral becomes

$$\begin{aligned}\iint_D (3 - x - y)\, dA &= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx \\ &= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx = \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{1} = 1\end{aligned}$$

Note that Methods 1 and 2 give the same answer. If they don't it means something is wrong.

0.11 Example

Evaluate

$$\iint_D (4x + 2)\, dA$$

where D is the region enclosed by the curves $y = x^2$ and $y = 2x$.
Solution. Again we will carry out the integration both ways, x first then y, and then vice versa, to ensure the same answer is obtained by both methods.

Method 1 : We do the integration first with respect to x and then with respect to y. We shall need to know where the two curves $y = x^2$ and $y = 2x$ intersect. They intersect when $x^2 = 2x$, i.e. when $x = 0, 2$. So they intersect at the points $(0, 0)$ and $(2, 4)$.

For a typical y, the horizontal line will enter D at $x = y/2$ and leave at $x = \sqrt{y}$. Then we need to let y go from 0 to 4 so that the horizontal line sweeps the entire region. Thus

$$\begin{aligned}\iint_D (4x + 2)\, dA &= \int_0^4 \int_{x=y/2}^{x=\sqrt{y}} (4x + 2)\, dx\, dy \\ &= \int_0^4 \left[2x^2 + 2x \right]_{x=y/2}^{x=\sqrt{y}} dy = \int_0^4 \left((2y + 2\sqrt{y}) - \left(\frac{y^2}{2} + y \right) \right) dy \\ &= \int_0^4 \left(y + 2y^{1/2} - \frac{y^2}{2} \right) dy = \left[\frac{y^2}{2} + \frac{2y^{3/2}}{3/2} - \frac{y^3}{6} \right]_0^4 = 8\end{aligned}$$

Method 2 : Integrate first with respect to y and then x, i.e. draw a vertical line across D at a typical x value. Such a line enters D at $y = x^2$ and leaves at $y = 2x$. The integral becomes

$$\begin{aligned}\iint_D (4x+2)\,dA &= \int_0^2\int_{x^2}^{2x} (4x+2)\,dy\,dx \\ &= \int_0^2 [4xy+2y]_{y=x^2}^{y=2x}\,dx \\ &= \int_0^2 \left(\left(8x^2+4x\right)-\left(4x^3+2x^2\right)\right)\,dx \\ &= \int_0^2 (6x^2-4x^3+4x)\,dx = \left[2x^3-x^4+2x^2\right]_0^2 = 8\end{aligned}$$

The example we have just done shows that it is sometimes easier to do it one way than the other. The next example shows that sometimes the difference in effort is more considerable. There is no general rule saying that one way is always easier than the other; it depends on the individual integral.

0.12 Example

Evaluate

$$\iint_D (xy-y^3)\,dA$$

where D is the region consisting of the square $\{(x,y):\ -1 \le x \le 0,\ 0 \le y \le 1\}$ together with the triangle $\{(x,y):\ x \le y \le 1,\ 0 \le x \le 1\}$.

Method 1 : (easy). integrate with respect to x first. A diagram will show that x goes from -1 to y, and then y goes from 0 to 1. The integral becomes

$$\begin{aligned}\iint_D (xy-y^3)\,dA &= \int_0^1\int_{-1}^{y} (xy-y^3)\,dx\,dy \\ &= \int_0^1 \left[\frac{x^2}{2}y - xy^3\right]_{x=-1}^{x=y} dy \\ &= \int_0^1 \left(\left(\frac{y^3}{2}-y^4\right)-(\tfrac{1}{2}y+y^3)\right)\,dy \\ &= \int_0^1 \left(-\frac{y^3}{2}-y^4-\tfrac{1}{2}y\right)\,dy = \left[-\frac{y^4}{8}-\frac{y^5}{5}-\frac{y^2}{4}\right]_{y=0}^{1} = -\frac{23}{40}\end{aligned}$$

Method 2 : (harder). It is necessary to break the region of integration D into two sub-regions D_1 (the square part) and D_2 (triangular part). The integral over D is given by

$$\iint_D (xy-y^3)\,dA = \iint_{D_1} (xy-y^3)\,dA + \iint_{D_2} (xy-y^3)\,dA$$

which is the analogy of the formula $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ for single integrals. Thus

$$\begin{aligned}
\iint_D (xy - y^3)\,dA &= \int_{-1}^{0}\int_0^1 (xy - y^3)\,dy\,dx + \int_0^1\int_x^1 (xy - y^3)\,dy\,dx \\
&= \int_{-1}^{0}\left[\frac{xy^2}{2} - \frac{y^4}{4}\right]_{y=0}^{1} dx + \int_0^1 \left[\frac{xy^2}{2} - \frac{y^4}{4}\right]_{y=x}^{1} dx \\
&= \int_{-1}^{0}\left(\tfrac{1}{2}x - \frac{1}{4}\right) dx + \int_0^1 \left(\left(\frac{x}{2} - \frac{1}{4}\right) - \left(\frac{x^3}{2} - \frac{x^4}{4}\right)\right) dx \\
&= \left[\frac{x^2}{4} - \frac{x}{4}\right]_{-1}^{0} + \left[\frac{x^2}{4} - \frac{x}{4} - \frac{x^4}{8} + \frac{x^5}{20}\right]_0^1 \\
&= -\tfrac{1}{2} - \frac{3}{40} = -\frac{23}{40}
\end{aligned}$$

In the next example the integration can only be done one way round.

0.13 Example

Evaluate

$$\iint_D \frac{\sin x}{x}\,dA$$

where D is the triangle $\{(x,y) :\ 0 \le y \le x,\ 0 \le x \le \pi\}$.
Solution. Let's try doing the integration first with respect to x and then y. This gives

$$\iint_D \frac{\sin x}{x}\,dA = \int_0^\pi \int_y^\pi \frac{\sin x}{x}\,dx\,dy$$

but we cannot proceed because we cannot find an indefinite integral for $\sin x/x$. So, let's try doing it the other way. We then have

$$\begin{aligned}
\iint_D \frac{\sin x}{x}\,dA &= \int_0^\pi \int_0^x \frac{\sin x}{x}\,dy\,dx \\
&= \int_0^\pi \left[\frac{\sin x}{x}\,y\right]_{y=0}^{x} dx = \int_0^\pi \sin x\,dx \\
&= [-\cos x]_0^\pi = 1 - (-1) = 2
\end{aligned}$$

0.14 Example

Find the volume of the tetrahedron that lies in the first octant and is bounded by the three coordinate planes and the plane $z = 5 - 2x - y$.
Solution. The given plane intersects the coordinate axes at the points $(\frac{5}{2}, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 5)$. Thus, we need to work out the double integral

$$\iint_D (5 - 2x - y)\,dA$$

where D is the triangle in the (x, y) plane with vertices $(x, y) = (0, 0)$, $(\frac{5}{2}, 0)$ and $(0, 5)$. It is a good idea to draw another diagram at this stage showing just the region D in the (x, y) plane. Note that the equation of the line joining the points $(\frac{5}{2}, 0)$ and $(0, 5)$ is $y = -2x + 5$. Then:

$$\begin{aligned}
\text{volume} &= \iint_D (5 - 2x - y)\, dA = \int_0^5 \int_0^{(5-y)/2} (5 - 2x - y)\, dx\, dy \\
&= \int_0^5 \left[5x - x^2 - yx\right]_{x=0}^{x=(5-y)/2} dy \\
&= \int_0^5 \left[5\left(\frac{5-y}{2}\right) - \left(\frac{5-y}{2}\right)^2 - y\left(\frac{5-y}{2}\right)\right] dy \\
&= \int_0^5 \left(\frac{25}{4} - \frac{5y}{2} + \frac{y^2}{4}\right) dy \\
&= \left[\frac{25y}{4} - \frac{5y^2}{4} + \frac{y^3}{12}\right]_0^5 = \frac{125}{12}
\end{aligned}$$

0.15 Changing variables in a double integral

We know how to change variables in a **single** integral:

$$\int_a^b f(x)\, dx = \int_A^B f(x(u)) \frac{dx}{du}\, du$$

where A and B are the new limits of integration.
For **double integrals** the rule is more complicated. Suppose we have

$$\iint_D f(x, y)\, dx\, dy$$

and want to change the variables to u and v given by $x = x(u, v)$, $y = y(u, v)$. The change of variables formula is

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v))|J|\, du\, dv \tag{0.1}$$

where J is the Jacobian, given by

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

and D^* is the new region of integration, in the (u, v) plane.

0.16 Transforming a double integral into polars

A very commonly used substitution is conversion into polars. This substitution is particularly suitable when the region of integration D is a circle or an annulus (i.e. region between two concentric circles). Polar coordinates r and θ are defined by

$$x = r\cos\theta, \quad y = r\sin\theta$$

The variables u and v in the general description above are r and θ in the polar coordinates context and the Jacobian for polar coordinates is

$$\begin{aligned} J &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) \\ &= r(\cos^2\theta + \sin^2\theta) = r \end{aligned}$$

So $|J| = r$ and the change of variables rule (0.1) becomes

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

0.17 Example

Use polar coordinates to evaluate

$$\iint_D xy\,dx\,dy$$

where D is the portion of the circle centre 0, radius 1, that lies in the first quadrant.
Solution. For the portion in the first quadrant we need $0 \le r \le 1$ and $0 \le \theta \le \pi/2$. These inequalities give us the limits of integration in the r and θ variables, and these limits will all be constants.
With $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\begin{aligned} \iint_D xy\,dx\,dy &= \int_0^{\pi/2}\int_0^1 r^2\cos\theta\sin\theta\,r\,dr\,d\theta \\ &= \int_0^{\pi/2}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=0}^1 d\theta \\ &= \int_0^{\pi/2}\frac{1}{4}\sin\theta\cos\theta\,d\theta = \int_0^{\pi/2}\frac{1}{8}\sin 2\theta\,d\theta \\ &= \frac{1}{8}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = \frac{1}{8} \end{aligned}$$

0.18 Example

Evaluate

$$\iint_D e^{-(x^2+y^2)}\,dx\,dy$$

where D is the region between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Solution. It is not feasible to attempt this integral by any method other than transforming into polars.
Let $x = r\cos\theta$, $y = r\sin\theta$. In terms of r and θ the region D between the two circles is described by $1 \le r \le 2$, $0 \le \theta \le 2\pi$, and so the integral becomes

$$\iint_D e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_1^2 e^{-r^2}\,r\,dr\,d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \left[-\tfrac{1}{2}e^{-r^2}\right]_{r=1}^{2} d\theta \\
&= \int_0^{2\pi} \left(-\tfrac{1}{2}e^{-4} + \tfrac{1}{2}e^{-1}\right) d\theta \\
&= \pi(e^{-1} - e^{-4})
\end{aligned}
$$

0.19 Example: integrating e^{-x^2}

The function e^{-x^2} has no elementary antiderivative. But we can evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ by using the theory of double integrals.

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) \\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\
&= \int_{-\infty}^{\infty} e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}
$$

Now transform to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The region of integration is the whole (x, y) plane. In polar variables this is given by $0 \le r < \infty$, $0 \le \theta \le 2\pi$. Thus

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \left[-\tfrac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} d\theta \\
&= \int_0^{2\pi} \tfrac{1}{2}d\theta = \pi
\end{aligned}
$$

We have shown that

$$
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi
$$

Hence

$$
\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.
$$

The above integral is very important in numerous applications.

0.20 Other substitutions

So far we have only illustrated how to convert a double integral into polars. We will now illustrate some examples of double integrals that can be evaluated by other substitutions. Unlike single integrals, for a double integral the choice of substitution is often dictated not only by what we have in the integrand but also by the shape of the region of integration.

0.21 Example

Evaluate

$$\iint_D (x+y)^2\, dx\, dy$$

where D is the parallelogram bounded by the lines $x+y=0$, $x+y=1$, $2x-y=0$ and $2x-y=3$.

Solution. (A diagram to show the region D will be useful).

In an example like this the boundary curves of D can suggest what substitution to use. So let us try

$$u = x+y, \qquad v = 2x-y.$$

In these new variables the region D is described by

$$0 \le u \le 1, \qquad 0 \le v \le 3.$$

We need to work out the Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

To work this out we need x and y in terms of u and v. From the equations $u = x+y$, $v = 2x-y$ we get

$$x = \frac{1}{3}(u+v), \qquad y = \frac{2}{3}u - \frac{1}{3}v$$

Therefore

$$J = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

and so $|J| = \frac{1}{3}$ (recall it is $|J|$ and not J that we put into the integral). Therefore the substitution formula gives

$$\iint_D (x+y)^2\, dx\, dy = \int_0^3\int_0^1 u^2 \underbrace{\frac{1}{3}}_{=|J|}\, du\, dv = \int_0^3 \left[\frac{u^3}{9}\right]_0^1 dv = \int_0^3 \frac{1}{9}\, dv = \frac{1}{3}.$$

0.22 Example

Let D be the region in the first quadrant bounded by the hyperbolas $xy=1$, $xy=9$ and the lines $y=x$, $y=4x$. Evaluate

$$\iint_D \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$$

Solution. A diagram showing D is useful. We make the substitution

$$xy = u^2, \qquad \frac{y}{x} = v^2.$$

We will need x and y in terms of u and v. By multiplying the above equations we get $y^2 = u^2v^2$. Hence $y = uv$ and $x = u/v$. In the (u, v) variables the region D is described by

$$1 \leq u \leq 3, \quad 1 \leq v \leq 2.$$

The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$$

Therefore

$$\begin{aligned} &\iint_D \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy \\ = &\iint (v+u)|J|\,du\,dv = \int_1^2\int_1^3 (v+u)\left(\frac{2u}{v}\right) du\,dv \\ = &\int_1^2\int_1^3 \left(2u + \frac{2u^2}{v}\right) du\,dv = \int_1^2 \left[u^2 + \frac{2u^3}{3v}\right]_{u=1}^{u=3} dv \\ = &\int_1^2 \left\{\left(9 + \frac{18}{v}\right) - \left(1 + \frac{2}{3v}\right)\right\} dv = \left[8v + \frac{52}{3}\ln v\right]_1^2 = 8 + \frac{52}{3}\ln 2. \end{aligned}$$

0.23 Application of double integrals: centres of gravity

We will show how double integrals may be used to find the location of the centre of gravity of a two-dimensional object. Mathematically speaking, a **plate** is a thin 2-dimensional distribution of matter considered as a subset of the (x, y) plane. Let

$$\sigma = \text{mass per unit area}$$

This is the definition of **density** for two-dimensional objects. If the plate is all made of the same material (a sheet of metal, perhaps) then σ would be a constant, the value of which would depend on the material of which the plate is made. However, if the plate is not all made of the same material then σ could vary from point to point on the plate and therefore be a function of x and y, $\sigma(x, y)$. For some objects, part of the object may be made of one material and part of it another (some currencies have coins that are like this). But $\sigma(x, y)$ could quite easily vary in a much more complicated way (a pizza is a simple example of an object with an uneven distribution of matter).

The intersection of the two thin strips defines a small rectangle of length δx and width δy. Thus

$$\begin{aligned} \text{mass of little rectangle} &= (\text{mass per unit area})(\text{area}) \\ &= \sigma(x, y)\,dx\,dy \end{aligned}$$

Therefore the total mass of the plate D is

$$M = \iint_D \sigma(x, y)\,dx\,dy.$$

Suppose you try to balance the plate D on a pin. The **centre of mass** of the plate is the point where you would need to put the pin. It can be shown that the coordinates

$(\bar{x}, \bar{y})$ of the centre of mass are given by

$$\bar{x} = \frac{\iint_D x\,\sigma(x,y)\,dA}{\iint_D \sigma(x,y)\,dA}, \qquad \bar{y} = \frac{\iint_D y\,\sigma(x,y)\,dA}{\iint_D \sigma(x,y)\,dA} \tag{0.2}$$

0.24 Example

A homogeneous triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$. Find the coordinates of its centre of mass.
[*'Homogeneous' means the plate is all made of the same material which is uniformly distributed across it, so that $\sigma(x,y) = \sigma$, a constant.*]
Solution. A diagram of the triangle would be useful. With σ constant, we have

$$\begin{aligned}\bar{x} &= \frac{\iint_D \sigma x\,dA}{\iint_D \sigma\,dA} = \frac{\sigma \int_0^1 \int_0^{3x} x\,dy\,dx}{\sigma \int_0^1 \int_0^{3x} dy\,dx} = \frac{\int_0^1 [xy]_{y=0}^{y=3x}\,dx}{\int_0^1 [y]_{y=0}^{y=3x}\,dx} \\ &= \frac{\int_0^1 3x^2\,dx}{\int_0^1 3x\,dx} = \frac{1}{3/2} = \frac{2}{3}\end{aligned}$$

and

$$\begin{aligned}\bar{y} &= \frac{\iint_D \sigma y\,dA}{\iint_D \sigma\,dA} = \frac{\sigma \int_0^1 \int_0^{3x} y\,dy\,dx}{\sigma \int_0^1 \int_0^{3x} dy\,dx} = \frac{\int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=3x} dx}{\int_0^1 [y]_{y=0}^{y=3x}\,dx} \\ &= \frac{\int_0^1 \frac{9x^2}{2}\,dx}{\int_0^1 3x\,dx} = \frac{3/2}{3/2} = 1.\end{aligned}$$

So the centre of mass is at $(\bar{x}, \bar{y}) = (\frac{2}{3}, 1)$.

0.25 Example

Find the centre of mass of a circle, centre the origin, radius 1, if the right half is made of material twice as heavy as the left half.
Solution. By symmetry, it is clear that the centre of mass will be somewhere on the x-axis, and so $\bar{y} = 0$. In order to model the fact that the right half is twice as heavy, we can take

$$\sigma(x,y) = \begin{cases} 2\sigma & x > 0 \\ \sigma & x < 0 \end{cases}$$

with the σ in the right hand side of the above expression being any positive constant.

From the general formula,

$$\bar{x} = \frac{\iint_D x\,\sigma(x,y)\,dA}{\iint_D \sigma(x,y)\,dA}. \tag{0.3}$$

Let us work out the integral in the numerator first. We shall need to break it up as follows

$$\iint_D x\sigma(x,y)\,dA = \iint_{\text{right half}} + \iint_{\text{left half}} = \iint_{\text{right}} 2\sigma x\,dA + \iint_{\text{left}} \sigma x\,dA$$

The circular geometry suggests we convert to plane polars, $x = r\cos\theta$, $y = r\sin\theta$. Recall that, in this coordinate system, $dA = r\,dr\,d\theta$. The right half of the circle is described by $-\pi/2 \le \theta \le \pi/2$, $0 \le r \le 1$, and the left half similarly but with $\pi/2 \le \theta \le 3\pi/2$. Thus

$$\begin{aligned}
\iint_D x\sigma(x,y)\,dA &= \int_{-\pi/2}^{\pi/2}\int_0^1 2\sigma(r\cos\theta)\,r\,dr\,d\theta + \int_{\pi/2}^{3\pi/2}\int_0^1 \sigma(r\cos\theta)\,r\,dr\,d\theta \\
&= 2\sigma\int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\cos\theta\right]_{r=0}^{r=1} d\theta + \sigma\int_{\pi/2}^{3\pi/2}\left[\frac{r^3}{3}\cos\theta\right]_{r=0}^{r=1} d\theta \\
&= \frac{2\sigma}{3}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta + \frac{\sigma}{3}\int_{\pi/2}^{3\pi/2}\cos\theta\,d\theta \\
&= \frac{4\sigma}{3} - \frac{2\sigma}{3} = \frac{2\sigma}{3}.
\end{aligned}$$

Finally, we work out the denominator in (0.3):

$$\begin{aligned}
\iint_D \sigma(x,y)\,dA &= \iint_{\text{left half}} \sigma\,dA + \iint_{\text{right half}} 2\sigma\,dA \\
&= \sigma\iint_{\text{left half}} dA + 2\sigma\iint_{\text{right half}} dA \\
&= \sigma(\text{area of left half}) + 2\sigma(\text{area of right half}) \\
&= \sigma(\pi/2) + 2\sigma(\pi/2) \\
&= \frac{3\sigma\pi}{2}
\end{aligned}$$

Therefore the x coordinate of the centre of mass of the object is

$$\bar{x} = \frac{2\sigma/3}{3\sigma\pi/2} = \frac{4}{9\pi}.$$

BETA AND GAMMA FUNCTIONS

GAMMA FUNCTION:

The integral of $\int_0^\infty e^{-x}x^{n-1}, dx(n>0)$ is a function of n .This is denoted by F(n) is known as Gamma function.

$$\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx,(n>0)$$

Properties of gamma functions:

(1.) $\Gamma(1)=1$

$$\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$$

Substitute n=1

$$\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$$

$$=\left(-e^{-x}\right)_0^\infty$$

$$=0-(-1)$$
$$=1$$
$$\therefore \Gamma(1)=1$$

(2.) $\Gamma n+1=n\Gamma n$

$$\Gamma n+1=\int_0^\infty e^{-x}x^{n}dx$$

$$u = x^n$$
$$du = nx^{n-1}$$
$$dv = e^{-x}$$
$$v = -e^x$$

$$= \left(-x^n e^{-x}\right)_0^\infty + n\int_0^\infty x^{n-1}e^{-x}dx$$

$$= 0 + n\int_0^\infty e^{-x}x^{n-1}dx$$

$$= n\Gamma n$$

$$\therefore \Gamma n+1 = n\Gamma n$$

(3.)When n is +ve integer

$$\Gamma n+1 = n!$$
$$\Gamma n+1 = n\Gamma n$$

$$= n.(n-1)\Gamma(n-1)$$
$$= n.(n-1)\text{.........................}2.1\Gamma 1$$
$$= n.(n-1)\text{.........................}2.1$$
$$= n!$$

(4.) $\Gamma n = 2\int_0^\infty e^{-t^2}t^{2n-1}dt$

Put $x = t^2$, $dx = 2xdt$ when $x \to 0, t \to 0$; $x \to \infty, t \to \infty$

$$\Gamma n = \int_0^\infty e^{-t}\left(t^2\right)^{n-1}.2tdt$$

$$= 2\int_0^\infty e^{-t^2}t^{2n-1}dt$$

$$\Gamma n = 2\int_0^\infty e^{-t^2}t^{2n-1}dt$$

BETA FUNCTION:

The beta function is defined as $\beta_{(m,n)} = \int_0^1 x^{m-1}(1-x)^{n-1} dx, m > 0, n > 0.$

Put $x = 1 - y$,

$$\beta_{(m,n)} = \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$\therefore \beta_{(m,n)} = \beta_{(n,m)}$$

Put $\begin{array}{l} x = \sin^2\theta \\ dx = 2\sin\theta\cos\theta d\theta \end{array}$

When $\begin{array}{l} x \to 0, \theta \to 0 \\ x \to 1, \theta \to \frac{\pi}{2} \end{array}$

$$\beta_{(m,n)} = \int_0^{\frac{\pi}{2}} \left(\sin^2\theta\right)^{m-1} \left(\cos^2\theta\right)^{n-1} 2\sin\theta\cos\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$$

RELATION BETWEEN BETA (β) AND GAMMA (Γ) FUNCTION:

$$\beta_{(m,n)} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

We know that $\Gamma m = \int_0^\infty e^{-t} t^{m-1} dt$

Put $\begin{array}{l} t = x^2 \\ dt = 2xdx \end{array}$

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1} dx$$

$$\Gamma n = 2\int_0^\infty e^{-x^2} y^{2n-1} dx$$

$$\therefore \Gamma m \Gamma n = 4\int_0^\infty e^{-x^2} x^{2m-1} dx \int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} x^{2m-1} y^{2n-1} dy$$

Changing polar co-ordinates $x = r\cos\theta, y = r\sin\theta$

$dxdy = rdrd\theta$

r varies from $0 to \infty$

θ varies from $0 to \dfrac{\pi}{2}$

$$= \Gamma m \Gamma n = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} \left(r\cos\theta\right)^{2m-1} \left(r\sin\theta\right)^{2n-1} rdrd\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta drd\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta . 2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr$$

$$\Gamma m \Gamma n = \beta_{(m,n)} . \Gamma m+n$$

$$\therefore \beta_{(m,n)} = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

Prove that $\int_0^\infty e^{-ax} x^{n-1} dx = \left[\Gamma n / a^n\right]$,when a and n are positive. Hence find the value of $\int_0^1 x^{q-1} \left[\log\left(1/x\right)\right]^{p-1} dx$.

Sol: we know that $\int_0^\infty e^{-ax} x^{n-1} dx = \Gamma n$

Put, ax = t

dx= dt/ a

$$\int_0^\infty e^{-ax} x^{n-1} dx = \int_0^\infty e^{-t} \left(t/a\right)^{n-1} \left(dt/a\right)$$

$$= 1/a^n \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \Gamma n / a^n$$

$$\int_0^1 x^{q-1} \left[\log\left(1/x\right)\right]^{p-1} dx = \int_\infty^0 e^{-(q-1)y} y^{p-1} \left(-e^{-y}\right) dy$$

Put, 1/x= e^y

$x = e^{-y} \to dx = -e^y dy$

When, $x \to 0, y \to \infty$
$x \to 1, y \to 0$

$= \int_0^\infty e^{-qy} y^{p-1} dy \quad = \left(1/q^p\right)\Gamma p$

2.prove that $\beta_{(m,n)} = \int_0^\infty \left(x^{m-1}/(1+x)^{m+n}\right)dx$. Hence deduce that

$\beta_{(m,n)} = \int_0^1 \left[x^{m-1} + x^{n-1}/(1+x)^{m+n}\right]dx$.

Sol: we know that $\beta_{(m,n)} = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put,x= t/1+t
dx=$1/(1+t)^2\, dt$
when, $x \to 0, t \to 0$
$x \to 1, t \to \infty$

$= \int_0^\infty (t/1+t)^{m-1} \quad (1/1+t)^{n-1}\left[1/(1+t)^2\right]dt$

$= \int_0^\infty t^{m-1}/(1+t)^{m+n}\, dt$

$= \int_0^1 t^{m-1}/(1+t)^{m+n} + \int_1^\infty t^{m-1}/(1+t)^{m+n}\, dt$

Consider , $\int_1^\infty t^{m-1}/(1+t)^{m+n}\, dt$

Put, t=1/y
Then, dt=-1/y 2 dy
When, $t \to 1, y \to 1$
$t \to \infty, y \to 0$

$= \int_0^1 y^{1/m-1}/(1+1/y)^{m+n}\left(-1/y^2\right)dy$

$= \int_0^1 y^{m+n}/(1+y)^{m+n} \quad y^{m+1} dy$

$= \int_0^1 y^{n-1}/(1+y)^{m+n} \quad dy$

$$= \int_0^1 t^{n-1} / (1+t)^{m+n}\, dt \qquad \text{[changing the domain variable]}$$

Then, $\int_0^1 t^{m-1} / (1+t)^{m+n}\, dt + \int_1^\infty t^{m-1} / (1+t)^{m+n}\, dt = \int_0^1 t^{m-1} / (1+t)^{m+n}\, dt + \int_0^1 t^{n-1} / (1+t)^{m+n}\, dt$

$$\beta_{(m,n)} = \int_0^1 t^{m-1} + t^{n-1} / (1+t)^{m+n}\, dt$$

3.Evaluate $\int_0^1 x^m (1-x^n)^p\, dx$ in terms of gamma functions and hence find $\int_0^1 dx / \sqrt{(1-x^n)}$.

Sol: $\int_0^1 x^m (1-x^n)^p\, dx$

Put, $x^n = t \rightarrow nx^{n-1}dx = dt \rightarrow dx = 1/n(dt/t^{1-1/n})$

When, $x \rightarrow 0, t \rightarrow 0$
$x \rightarrow 1, t \rightarrow 1$

$$= \int_0^1 t^{m/n} (1-t)^p 1/n(t)^{1-n/n}\, dt$$

$$= 1/\text{n} \int_0^1 t^{m-n+1/n} (1-t)^p\, dt$$

$$= 1/\text{n}\, \beta_{(m+1/n,\, p+1)}$$

$= 1/\text{n}\, \Gamma(m+1/n)\Gamma(p+1) / \Gamma(m+1/n+p+1)$ $\int_0^1 dx / \sqrt{(1-x^n)} = \int_0^1 x^0 (1-x^n)^{-1/2}\, dx$

Here, m=0, n=n, p=-1/2

$$\therefore \int_0^1 dx / \sqrt{(1-x^n)} = 1/n\left[\Gamma(1/n)\Gamma(1/2)\right] / \Gamma(1/n+1/2)$$

$$= \sqrt{(\pi)} / n\Gamma(1/n) / \Gamma(n+2/2n)$$

4. prove that $\int_0^\infty x^4 . e^{-x^2}\, dx$

Sol: $\int_0^\infty x^4 . e^{-x^2}\, dx$

$$= \int_0^\infty x^4 . e^{-x^2}\, dx$$

$$= \quad 1/2.\int_0^\infty t^{3/2}.e^{-t}dt \quad \begin{aligned} x^2 &= t \\ 2xdx &= dt \\ dx &= \frac{dt}{2\sqrt{t}} \end{aligned}$$

$$= \quad \Gamma n = \int_0^\infty e^{-x}.x^{n-1}dx$$

$$1/2\int_0^\infty t^{3/2}.e^{-t}dt = \frac{1}{2}.\Gamma\frac{5}{2}$$

when x=o, $t = \infty$

$$= \frac{1}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2} \quad x = \infty, t = \infty$$

$$= \quad \frac{3}{8}\sqrt{\pi}$$

n-1=3/2

$$\int_0^{\pi/2} \sin^3 x.\cos^{5/2} xdx$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta.d\theta$$

$$I = \frac{1}{2}\beta(2,7/4)$$

$$I = \frac{1}{2}\frac{\Gamma 2.\Gamma 7/2}{\Gamma 2 + 7/4}$$

2m-1=3,

2m=4,

M=2.

2n-1=5/2,

2n=7/2,

n=7/4

$$I = \frac{1!.\frac{3}{4}.\Gamma\frac{3}{4}}{\Gamma\frac{15}{4}}$$

$$I = \frac{3}{8}.\frac{\Gamma\frac{3}{4}}{\frac{77}{16}.\frac{3}{4}.\Gamma\frac{3}{4}}$$

$$\Gamma\frac{15}{4} = \frac{11}{4}\Gamma\frac{11}{4}$$

$$= \frac{11}{4}.\frac{7}{4}\Gamma\frac{7}{4}$$

$$I = \frac{3}{8}.\frac{16}{77}.\frac{4}{3}$$

$$I = \frac{8}{77}$$

$$= \frac{77}{16}.\frac{3}{4}\Gamma\frac{3}{4}$$

5.

$$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta.\cos^{-1/2}\theta.d\theta$$

$$= \frac{1}{2}\beta(\frac{3}{4,}\frac{1}{4})$$

$$= \frac{1}{2}\frac{\Gamma\frac{3}{4}.\Gamma\frac{1}{4}}{\Gamma 1}$$

$$= \frac{1}{2}\Gamma\frac{3}{4}.\Gamma\frac{1}{4}$$

$2m-1=1/2$ $2n-1=-1/2$

$2m=3/2$ $2n=1/2$

$m=3/4$ $n=1/4$

$$9. \int_0^{\pi/2} \sqrt{\cot\theta}\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\cos\theta}{\sin\theta}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \cos^{1/2}\theta\sin^{-1/2}\theta d\theta$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \cos^{1/2}\theta\sin^{-1/2}\theta d\theta$$

$$= \frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right)$$

$$= \frac{1}{2}\left|\frac{1}{2}\right|\frac{3}{4}$$

2m-1=-1/2	2n-1=1/2
M=1/4	n=3/4

10.prove that

$$\int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx = \lceil n$$

$$= \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx$$

$$= -\int_\infty^0 y^{n-1} e^{-y} dy$$

$$= \int_0^\infty e^{-y} y^{n-1} dy$$

$$= \lceil n$$

put $\log\left(\frac{1}{x}\right) = y \quad \Rightarrow \frac{1}{x} = e^y$

$e^{-y} = x$

$dx = -e^{-y} dy$

when

$x \to 0, y \to \infty$

$x \to 1, y \to 0$

11.prove that

$$\beta(m+1,n) + \beta(m,n+1) = \beta(m,n)$$

$$\beta(m+1,n) + \beta(m,n+1) = \frac{\Gamma m+1.\Gamma n}{\Gamma m+n+1} + \frac{\Gamma m \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{m\Gamma m\Gamma n + \Gamma m.n\Gamma n}{\Gamma m+n+1}$$

$$= \frac{\Gamma m\Gamma n(m+n)}{(m+n)\Gamma m+n}$$

$$= \beta(m,n)$$

12.prove that $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{4\sqrt{2}}$

$I_1 = \int_0^\infty e^{-x^2} x^{-1/2} dx \quad I_2 = \int_0^\infty x^2 e^{-x^4} dx$

$t = x^4$

$t = x^2 \quad x = t^{1/2} \quad x = t^{1/4}$

dt=2xdx

dx= $\frac{dt}{2x} = \frac{dt}{2t^{1/2}}$

$dt = 4x^3 dx$

$$dx = \frac{dx}{4x^3} = \frac{dt}{4t^{3/4}}$$

$$= \int_0^\infty \left(t^{1/4}\right)^2 \; e^{-t} \frac{dt}{4t^{3/4}}$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{1}{2}-\frac{3}{4}} dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4} dt$$

$$n-1 = -\frac{1}{4}$$

$$n = \frac{3}{4}$$

$$= \frac{1}{4}\Gamma 3/4$$

$$=\int_0^\infty e^{-t}\left(t^{-1/2}\right)^{1/2}\frac{dt}{2t^{1/2}}$$

$$=\frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{4}-\frac{1}{2}}dt$$

$$=\frac{1}{2}\int_0^\infty e^{-t}t^{-3/4}dt$$

n-1=-3/4

n=1-3/4

n=1/4

$$=\frac{1}{4}\Gamma 1/4$$

$$I_1=\frac{1}{2}\Gamma 1/2$$

$$I_2=\frac{1}{4}\Gamma 3/4$$

$$w.k.t$$

$$=\frac{1}{2^{2n-1}}\beta\left(n,\frac{1}{2}\right)$$

$$(i.e)\frac{\Gamma n\Gamma n}{\Gamma 2n}=\frac{1}{2^{2n-1}}\frac{\Gamma n\Gamma 1/2}{\Gamma n+1/2}$$

$$\Gamma n\Gamma n+1/2=\frac{\sqrt{\pi}\Gamma 1/2}{2^{-1/2}}$$

$=\pi\sqrt{2}$

n=1/4

$$\left|\frac{1}{4}\right|\frac{2}{4}=\frac{\sqrt{\pi}\left|\frac{1}{2}\right.}{2^{-1/2}}$$

$=\pi\sqrt{2}$

Using this in above

$$\int_0^\infty\frac{e^{-x^2}}{\sqrt{x}}dx\int_0^\infty x^2e^{-x^4}dx=\frac{\pi\sqrt{2}}{8}$$

$$= \frac{\pi}{4\sqrt{2}}$$

13. *Evaluate* $\int_0^\infty \frac{x^{m-1}}{(1+x^n)p} dx$ and deduce that $\int_0^\infty \frac{x^{m-1}}{1+x^n} dx = \frac{\pi}{n\sin(\frac{m\pi}{n})}$

Then show that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Sol:-

$$\int_0^1 \frac{x^{m-1}}{\left(1+x^n\right)p} dx$$

$$= \int_0^1 \frac{t^{-(m-1/n)}.(1-t)^{m-1/n}}{t^{-p}} \cdot \frac{dt}{nt^{-2(n-1/n)}.(1-t)^{n-1/n}}$$

$$= 1/n. \int_0^1 t^{p-m/n-1}.(1-t)^{m/n-1}.dt$$

$$= 1/n. \beta(p-m/n, m/n)$$

$$= 1/n. \frac{\Gamma p - m/n \Gamma m/n}{\Gamma p}$$

Put p=1

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{4} \cos ec\pi/4$$

$$= \frac{\pi}{2\sqrt{2}}$$

14. *Evaluate* $\int_0^{\pi/2} \sin^5\theta.\cos^7\theta.d\theta$

$$\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta.d\theta$$

$$I = 1/2.\beta(2, 7/4)$$

$$= 1/2.\frac{\Gamma 2.\Gamma 7/4}{\Gamma 2 + 7/4}$$

$$\Gamma n+1 = n\Gamma n$$
$$2m-1 = 3$$
$$2m = 4$$
$$m = 2$$

$$= 1/2.\frac{1!.3/4.\Gamma 3/4}{\Gamma 15/4}$$

$$2n-1 = 5/2$$
$$2n = 7/2$$
$$n = 7/4$$

$$= 3/8.\frac{\Gamma 3/4}{77/16.3/4.\Gamma 3/4}$$

$$= 3/8.16/77.4/3$$

$$= 8/77$$

$$\Gamma 15/4 = 11/4$$
$$= 11/4.7/4.\Gamma 7/4$$
$$= 77/16.3/4\Gamma 3/4$$

15. Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}.d\theta$

$$\int_0^{\pi/2} \sqrt{\tan\theta}.d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta.\cos^{-1/2}\theta\, d\theta$$

$$2m-1 = 1/2$$
$$2m = 3/2$$
$$m = 3/4$$

$$= 1/2.\beta(3/4, 1/4)$$

$$= 1/2.\frac{\Gamma 3/4.\Gamma 1/4}{\Gamma 1}$$

$$= 1/2.\Gamma 3/4.\Gamma 1/4$$

$$2n-1 = -1/2$$
$$2n = 1/2$$
$$n = 1/4$$

16. Prove that (i) $\beta_{(m,\frac{1}{2})} = 2^{2m-1}\beta_{(m,m)}$

(ii) $\Gamma_m \Gamma_{(m,\frac{1}{2})} = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma_{2m}$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta.d\theta$$

$$\beta(m,\frac{1}{2}) = \int_0^{\pi/2} \sin^{2m-1}\theta d\theta$$

$$\beta(m,m) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-1} 2\theta d\theta$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1}\Phi d\Phi$$

$$= 2^{2m-1}.\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} d\theta = \beta(m,1/2)$$

from(*ii*)

$$2^{2m-1}\beta(m,n) = \beta(m,1/2)$$

$$2^{2m-1}\frac{\Gamma m \Gamma n}{\Gamma m+n} = \frac{\Gamma m \Gamma 1/2}{\Gamma m+1/2}$$

$$\Gamma m \Gamma m+1/2 = \frac{\Gamma m \Gamma 2m}{\Gamma m+1/2}$$

$$\Gamma m \Gamma m+1/2 = \frac{\Gamma m \Gamma 2m}{2^{2m-1}}$$

FOURIER SERIES

Particular Cases

Case (i)

If f(x) is defined over the interval (0,2*l*).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)xdx, \qquad n = 1,2,.....\infty$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)xdx,$$

If f(x) is defined over the interval (0,2π).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$, n=1,2,.....∞

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$ n=1,2,.....∞

Case (ii)

If f(x) is defined over the interval (-*l* , *l*).

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}]$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x) dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right) x dx$$

n =1,2,...... ∞

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right) x dx,$$

n=1,2,...... ∞

If f(x) is defined over the interval (-π, π).

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

a_0 = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$, n=1,2,.....∞

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$ n=1,2,.....∞

Problem: Obtain the Fourier expansion of

$$f(x) = \frac{1}{2}(\pi - x) \text{ in } -\pi < x < \pi$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{-\pi}^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\cos nx dx$$

Here we use integration by parts, so that

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{n}$$

Using the values of a_0 , a_n and b_n in the Fourier expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we get,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

This is the required Fourier expansion of the given function.

Problem: Obtain the Fourier expansion of f(x)=e^{-ax} in the interval (-π, π). Deduce that

$$\operatorname{cosech}\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

Here, $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left\{-a\cos nx + n\sin nx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left\{-a\sin nx - n\cos nx\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

Thus,

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For x=0, a=1, the series reduces to

$$f(0)=1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

or

$$1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

or $$1 = \frac{2\sinh \pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

Thus,

$$\pi \cos ech\pi = 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

This is the desired deduction.

Problem: Obtain the Fourier expansion of f(x) = x^2 over the interval ($-\pi$, π). Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...+\infty$$

Solution:

The function f(x) is even. Hence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

or
$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx, \quad \text{since f(x)cosnx is even}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx$$

Integrating by parts, we get

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

Also, $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$ since f(x)sinnx is odd.

Thus

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..$$

Problem: Obtain the Fourier expansion of

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Solution:

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} xdx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$ since f(x)cosnx is even.

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nxdx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0,$$ since f(x)sinnx is odd

Thus the Fourier series of f(x) is

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n - 1\right\}\cos nx$$

For x=π , we get

$$f(\pi) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n - 1\right\}\cos n\pi$$

or $$\pi = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2\cos(2n-1)\pi}{(2n-1)^2}$$

Thus,

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

This is the series as required.

Problem: Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$$

Deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Solution:

Here,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi dx + \int_{0}^{\pi} x dx\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx dx + \int_{0}^{\pi} x\cos nx dx\right]$$

$$= \frac{1}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx dx + \int_{0}^{\pi} x\sin nx dx\right]$$

$$= \frac{1}{n}\left\{1 - 2(-1)^n\right\}$$

Fourier series is

$$f(x) = \frac{-\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n - 1\right\}\cos nx + \sum_{n=1}^{\infty}\frac{\left\{1 - 2(-1)^n\right\}}{n}\sin nx$$

Note that the point x=0 is a point of discontinuity of f(x). Here $f(x^+)=0$, $f(x^-)=-\pi$ at x=0. Hence

$$\frac{1}{2}[f(x^+)+f(x^-)]=\frac{1}{2}(0-\pi)=\frac{-\pi}{2}$$

The Fourier expansion of f(x) at x=0 becomes

$$\frac{-\pi}{2}=\frac{-\pi}{4}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n-1]$$

$$or\ \frac{\pi^2}{4}=\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n-1]$$

Simplifying we get, $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$

Problem: Obtain the Fourier series of $f(x) = 1-x^2$ over the interval (-1,1).

Solution:

The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here

$$a_0=\frac{1}{1}\int_{-1}^{1}f(x)dx=2\int_{0}^{1}f(x)dx$$

$$=2\int_{0}^{1}(1-x^2)dx=2\left[x-\frac{x^3}{3}\right]_0^1$$

$$=\frac{4}{3}$$

$$a_n=\frac{1}{1}\int_{-1}^{1}f(x)\cos(n\pi x)dx$$

$$=2\int_{0}^{1}f(x)\cos(n\pi x)dx$$ as $f(x)\cos(n\pi x)$ is even

$$=2\int_{0}^{1}(1-x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n=2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)-(-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)+(-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$=\frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n=\frac{1}{1}\int_{-1}^{1}f(x)\sin(n\pi x)dx$$ =0, since $f(x)\sin(n\pi x)$ is odd.

The Fourier series of f(x) is $f(x)=\frac{2}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$

Problem: Obtain the Fourier expansion of

f(x) = $\begin{cases} 1+\dfrac{4x}{3}, & -\dfrac{3}{2} < x \le 0 \\ 1-\dfrac{4x}{3}, & 0 \le x < \dfrac{3}{2} \end{cases}$

Deduce that $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots$

Solution:

The period of f(x) is $\dfrac{3}{2} - \left(\dfrac{-3}{2}\right) = 3$

Also f(-x) = f(x). Hence f(x) is even

$$a_0 = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)dx = \frac{2}{3/2}\int_0^{3/2} f(x)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx = 0$$

$$a_n = \frac{1}{3/2}\int_{-3/2}^{3/2} f(x)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left[\left(1-\frac{4x}{3}\right)\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)}\right) - \left(\frac{-4}{3}\right)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2}\right)\right]_0^{3/2}$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Also,

$$b_n = \frac{1}{3}\int_{-3/2}^{3/2} f(x)\sin\left(\frac{n\pi x}{3/2}\right)dx = 0$$

Thus

f(x) = $\dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{2n\pi x}{3}\right)$

putting x=0, we get

f(0) = $\dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]$

or $$1 = \frac{8}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\right]$$

Thus, $$\frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$$

HALF-RANGE FOURIER SERIES

The Fourier expansion of the periodic function f(x) of period 2*l* may contain both sine and cosine terms. Many a time it is required to obtain the Fourier expansion of f(x) in the interval (0,*l*) which is regarded as half interval. The definition can be extended to the other half in such a manner that the function becomes even or odd. This will result in cosine series or sine series only.

Sine series :

Suppose f(x) = φ(x) is given in the interval (0,*l*). Then we define f(x) = -φ(-x) in (-*l*,0). Hence f(x) becomes an odd function in (-*l* , *l*). The Fourier series then is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \qquad (11)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

The series (11) is called half-range sine series over (0,*l*).

Putting l=π in (11), we obtain the half-range sine series of f(x) over (0,π) given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Cosine series :

Let us define

$$f(x) = \begin{cases} \phi(x) & \text{in } (0,l) \quad \ldots\text{given} \\ \phi(-x) & \text{in } (-l,0) \quad \ldots\text{in order to make the function even.} \end{cases}$$

Then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad (12)$$

where,
$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

The series (12) is called half-range cosine series over (0,*l*)

Putting l = π in (12), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \quad \mathrm{n} = 1,2,3,\ldots..$$

Problem: Expand f(x) = x(π-x) as half-range sine series over the interval (0,π).
Solution: We have,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx dx$$

Integrating by parts, we get

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

The sine series of f(x) is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[1 - (-1)^n\right]\sin nx$$

Problem: Obtain the cosine series of

$$f(x) = \begin{cases} x, 0 < x < \frac{\pi}{2} \\ \pi - x, \frac{\pi}{2} < x < \pi \end{cases} \qquad over(0,\pi)$$

Solution:

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi} (\pi - x)dx\right] = \frac{\pi}{2}$$

Here

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx dx\right]$$

Performing integration by parts and simplifying, we get

$$a_n = -\frac{2}{n^2\pi}\left[1+(-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{8}{n^2\pi}, n = 2,6,10,.....$$

Thus, the Fourier cosine series is

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + ...\infty\right]$$

Problem: Obtain the half-range cosine series of f(x) = c-x in 0<x<c

Solution:

Here

$$a_0 = \frac{2}{c}\int_0^c (c-x)dx = c$$

$$a_n = \frac{2}{c}\int_0^c (c-x)\cos\left(\frac{n\pi x}{c}\right)dx$$

Integrating by parts and simplifying we get,

$$a_n = \frac{2c}{n^2\pi^2}\left[1-(-1)^n\right]$$

The cosine series is given by

$$f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{c}\right)$$

FOURIER TRANSFORMS

Introduction

The Fourier series expresses any periodic function into a sum of sinusoids. The Fourier transform is the extension of this idea to non-periodic functions by taking the limiting form of Fourier series when the fundamental period is made very large (infinite). Fourier transform finds its applications in astronomy, signal processing, linear time invariant (LTI) systems etc.

<u>Some useful results in computation of the Fourier transforms:</u>

1. $\int_0^\infty e^{-ax} \sin \lambda x \, dx = \frac{\lambda}{a^2+\lambda^2}$
2. $\int_0^\infty e^{-ax} \cos \lambda x \, dx = \frac{a}{a^2+\lambda^2}$
3. $\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}, \lambda > 0$

   When $\lambda = 1$, $\int_0^\infty \frac{sinx}{x} dx = \frac{\pi}{2}$
4. $\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$
5. $\cos ax = \frac{e^{iax} + e^{-iax}}{2}$
6. $\int_0^\infty e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}$

   When $a = 1$, $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$
7. Heaviside Step Function or Unit step function $H(t)$or $U(t) = \begin{cases} 0, \text{ when } t < 0 \\ 1, \text{when } t \geq 0 \end{cases}$

   At $t = 0$, $H(t)$ is sometimes taken as 0.5 or it may not have any specific value.

   Shifting at $t = a$

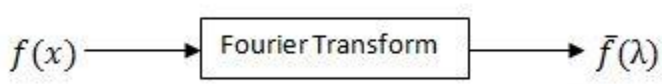
   $H(t-a)$ or $U(t-a) = \begin{cases} 0, \text{ when } t < a \\ 1, \text{when } t \geq a \end{cases}$
8. Dirac Delta Function or Unit Impulse Function is defined as $\delta(t-a) = 0$, $t \neq a$ *such* that $\int_0^\infty \delta(t-a)dt = 1$, $a \geq 0$. It is zero everywhere except one point 'a'. Delta function in sometimes thought of having infinite value at $t = a$. The delta function can be viewed as the derivative of the Heaviside step function

Dirichlet's Conditions for Existence of Fourier Transform

Fourier transform can be applied to any function $f(x)$ if it satisfies the following conditions:

1. $f(x)$ is absolutely integrable i.e. $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent.
2. The function $f(x)$ has a finite number of maxima and minima.
3. $f(x)$ has only a finite number of discontinuities in any finite

Fourier Transform, Inverse Fourier Transform and Fourier Integral



The Fourier transform of $f(x), -\infty < x < \infty$, denoted by $\bar{f}(\lambda)$ where $\lambda \in N$, is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx \quad \ldots ①$$

Also inverse Fourier transform of $\bar{f}(\lambda)$ gives $f(x)$ as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x}\, \bar{f}(\lambda) d\lambda \ldots ②$$

Rewriting ① as $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} f(t) dt$ and using in ②, Fourier integral representation of $f(x)$ is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda(t-x)}\, f(t) dt\, d\lambda$$

Fourier Sine Transform (F.S.T.)

Fourier Sine transform of $f(x), 0 < x < \infty$, denoted by $\bar{f}_s(\lambda)$, is given by

$$F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \lambda x\, dx \ldots ③$$

Also inverse Fourier Sine transform of $\bar{f}_s(\lambda)$ gives $f(x)$ as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_s(\lambda) \sin \lambda x\, d\lambda \ldots ④$$

Rewriting ③ as $\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \lambda t\, dt$ and using in ④, Fourier sine integral representation of $f(x)$ is given by:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin \lambda t \sin \lambda x\, dt d\lambda$$

2.2.2 Fourier Cosine Transform (F.C.T.)

Fourier Cosine transform of $f(x), 0 < x < \infty$, denoted by $\bar{f}_c(\lambda)$, is given by

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x \, dx \ldots ⑤$$

Also inverse Fourier Cosine transform of $\bar{f}_c(\lambda)$ gives $f(x)$ as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda x \, d\lambda \ldots ⑥$$

Rewriting ⑤ as $\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t \, dt$ and using in ⑥, Fourier cosine integral representation of $f(x)$ is given by:

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos \lambda t \cos \lambda x \, dt d\lambda$$

Remark:

- Parameter λ may be taken as *p*, *s* or ω as per usual notations.
- Fourier transform of $f(x)$ may be given by $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) dx$,

  then Inverse Fourier transform of $\bar{f}(\lambda)$is given by $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} \bar{f}(\lambda) d\lambda$
- Sometimes Fourier transform of $f(x)$ is taken as $\bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx$,

  thereby Inverse Fourier transform is given by $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \bar{f}(\lambda) d\lambda$

  Similarly if Fourier Sine transform is taken as $\bar{f}_s(\lambda) = \int_0^\infty f(x) \sin \lambda x \, dx$,

  then Inverse Sine transform is given by $f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x \, d\lambda$

  Similar is the case with Fourier Cosine transform.

Example 1 State giving reasons whether the Fourier transforms of the following functions exist: i. $\sin \frac{1}{x}$ ii.e^x iii. $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

Solution: i. The graph of $\sin \frac{1}{x}$ oscillates infinite number of times at $x = n\pi, n \in Z$

$\therefore f(x) \sin \frac{1}{x}$ is having infinite number of maxima and minima in the interval $(-\infty, \infty)$. Hence Fourier transform of $f(x) = \sin \frac{1}{x}$ does not exist.

ii. For $f(x) = e^x$, $\int_{-\infty}^{\infty} |e^x| dx$ is not convergent. Hence Fourier transform of e^x does not exist.

iii. $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is having infinite number of maxima and minima in the interval $(-\infty, \infty)$. Hence Fourier transform of $f(x)$ does not exist.

Example 2 Find Fourier Sine transform of

i. $\frac{1}{x}$ ii. $2e^{-3x} + 3e^{-2x}$

Solution: i. By definition, we have $F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin \lambda x \, dx = \sqrt{\frac{2}{\pi}} . \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

ii. By definition, $F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty 2e^{-3x} \sin \lambda x \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty 3e^{-2x} \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2e^{-3x}}{9+\lambda^2} (-3 \sin \lambda x - \lambda \cos \lambda x \right]_0^\infty + \sqrt{\frac{2}{\pi}} \left[\frac{3e^{-2x}}{4+\lambda^2} (-2 \sin \lambda x - \lambda \cos \lambda x \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 + \frac{2\lambda}{9+\lambda^2} \right] + \sqrt{\frac{2}{\pi}} \left[0 + \frac{3\lambda}{4+\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\lambda}{9+\lambda^2} + \frac{3\lambda}{4+\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{5\lambda^3+35\lambda}{(9+\lambda^2)(4+\lambda^2)} \right]$$

Example 3 Find Fourier transform of Delta function $\delta(x-a)$

Solution: $F\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\lambda x} . \delta(x-a) \, dx$

$$= \frac{1}{\sqrt{2\pi}} e^{i\lambda a}$$

$\because \int_{-\infty}^\infty f(t) \ \delta(t-a) dt = f(a)$ by virtue of fundamental property of Delta function where $f(t)$ is any differentiable function.

Example 4 Show that Fourier sine and cosine transforms of x^{n-1} are $\frac{\lceil n}{\lambda^n} \sin \frac{n\pi}{2}$ and $\frac{\lceil n}{\lambda^n} \cos \frac{n\pi}{2}$ respectively.

Solution: By definition, $\int_0^\infty e^{-t} \, t^{n-1} dt = \lceil n$

Putting $t = i\lambda x$ so that $dt = i\lambda dx$

$$\Rightarrow \int_0^\infty e^{-i\lambda x} \, (i\lambda x)^{n-1} i\lambda \, dx = \lceil n$$

$$\Rightarrow \int_0^\infty x^{n-1} e^{-i\lambda x}\, dx = \frac{\lceil n \; i^{-n}}{\lambda^n}$$

$$\Rightarrow \int_0^\infty x^{n-1} (\cos \lambda x - i \sin \lambda x)\, dx = \frac{\lceil n}{\lambda^n}\left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right)$$

$$\because i^{-n} = \left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right)$$

$$\Rightarrow \int_0^\infty x^{n-1} \cos \lambda x\, dx - i \int_0^\infty x^{n-1} \sin \lambda x\, dx = \frac{\lceil n}{\lambda^n}\cos\frac{n\pi}{2} - i\, \frac{\lceil n}{\lambda^n}\sin\frac{n\pi}{2}$$

Equating real and imaginary parts, we get

$$\int_0^\infty x^{n-1} \cos \lambda x\, dx = \frac{\lceil n}{\lambda^n}\cos\frac{n\pi}{2} \text{ and } \int_0^\infty x^{n-1} \sin \lambda x\, dx = \frac{\lceil n}{\lambda^n}\sin\frac{n\pi}{2}$$

$$\Rightarrow \bar{f}_c(\lambda) = \frac{\lceil n}{\lambda^n}\cos\frac{n\pi}{2} \text{ and } \bar{f}_s(\lambda) = \frac{\lceil n}{\lambda^n}\sin\frac{n\pi}{2}$$

Example 5 Find Fourier Cosine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

Solution: By definition, we have $F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x\, dx$

$$\therefore \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos \lambda x\, dx + \int_1^2 (2 - x) \cos \lambda x\, dx + \int_2^\infty 0. \cos \lambda x\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[(x)\left(\frac{\sin \lambda x}{\lambda}\right) - (1)\left(-\frac{\cos \lambda x}{\lambda^2}\right)\right]_0^1 + \left[(2 - x)\left(\frac{\sin \lambda x}{\lambda}\right) - (-1)\left(-\frac{\cos \lambda x}{\lambda^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2} - \frac{1}{\lambda^2} - \frac{\cos 2\lambda}{\lambda^2} - \frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\cos \lambda - \cos 2\lambda - 1}{\lambda^2}\right]$$

Example 6 Find Fourier Sine and Cosine transform of $f(x) = e^{-x}$ and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$$

Solution: To find Fourier Sine transform

$$F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x\, dx$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \lambda x\, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{1+\lambda^2}\right) \ldots\ldots ①$$

Taking inverse Fourier Sine transform of ①

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x\, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2} sin\lambda x d\lambda \text{.....②}$$

Substituting $f(x) = e^{-x}$ in ②

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda sin\lambda x}{1+\lambda^2} d\lambda$$

Replacing x by m on both sides

$$\Rightarrow e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{\lambda sin\lambda m}{1+\lambda^2} d\lambda$$

Now by property of definite integrals $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2} dx \text{ ....③}$$

Similarly taking Fourier Cosine transform of $f(x) = e^{-x}$

$$F_c\{f(x)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \cos \lambda x\, dx$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x} \cos \lambda x\, dx = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\lambda^2}\right) \text{......④}$$

Taking inverse Fourier Cosine transform of ④

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f_c}(\lambda) \cos \lambda x\, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2} \cos \lambda x\, d\lambda \text{..... ⑤}$$

Substituting $f(x) = e^{-x}$ in ⑤

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda$$

Replacing x by m on both sides

$$\Rightarrow e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{\cos \lambda m}{1+\lambda^2} d\lambda$$

Again by property of definite integrals $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{\cos mx}{1+x^2} dx \text{ .... ⑥}$$

From ③and ⑥, we get

$$\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2} dx$$

Example 7 Find Fourier transform of $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

and hence evaluate $\int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx$

Solution: Fourier transform of $f(x)$ is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{i\lambda x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\left(\frac{e^{i\lambda x}}{i\lambda}\right)-(-2x)\left(\frac{e^{i\lambda x}}{i^2\lambda^2}\right)+(-2)\left(\frac{e^{i\lambda x}}{i^3\lambda^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2e^{i\lambda}}{i^2\lambda^2}-\frac{2e^{i\lambda}}{i^3\lambda^3}+\frac{2e^{-i\lambda}}{i^2\lambda^2}+\frac{2e^{-i\lambda}}{i^3\lambda^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{e^{i\lambda}+e^{-i\lambda}}{\lambda^2}+\frac{e^{i\lambda}-e^{-i\lambda}}{i\lambda^3}\right] \qquad \because i^2=-1 \text{ and } i^3=-i$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2\cos\lambda}{\lambda^2}+\frac{2\sin\lambda}{\lambda^3}\right]$$

$$\therefore \bar{f}(\lambda) = \frac{2\sqrt{2}}{\sqrt{\pi}}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right) \ .....①$$

Taking inverse Fourier transform of ①

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}\,\bar{f}(\lambda)d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}(\cos\lambda x - i\sin\lambda x)\left(\frac{\lambda\cos\lambda-\sin\lambda}{\lambda^3}\right)d\lambda \ \because e^{-i\lambda x}=\cos\lambda x - i\sin\lambda x$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\cos\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)-i\sin\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)\right]d\lambda ....②$$

Substituting $f(x) = \begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases}$ in ②

$$\Rightarrow \begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\cos\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)-i\sin\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)\right]d\lambda$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty}\cos\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda = \begin{cases}\frac{\pi}{2}(1-x^2), & |x|<1\\ 0, & |x|>1\end{cases}$$

Putting $x=\frac{1}{2}$ on both sides

$$\int_{-\infty}^{\infty}\cos\frac{\lambda}{2}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda = \frac{\pi}{2}\left(1-\frac{1}{4}\right)$$

$$\Rightarrow 2\int_{0}^{\infty}\cos\frac{\lambda}{2}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda = \frac{3\pi}{8} \qquad \because \cos\frac{\lambda}{2}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right) \text{ is an even function of } \lambda$$

Now by property of definite integrals $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$$

Example 8 Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$

Solution: To find Fourier cosine transform

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx$$

$$\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx \;\ldots\ldots ①$$

To evaluate the integral given by ①

Let $g(x) = e^{-x}$ …… ②

$$F_c\{g(x)\} \equiv \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos\lambda x\,dx$$

$$\Rightarrow \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\lambda x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$\Rightarrow \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}$$

Again taking Inverse Fourier cosine transform

$$g(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda$$

$$\Rightarrow g(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx$$

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx = \frac{\pi}{2}\,g(\lambda) \quad \ldots\ldots ③$$

Using ② in ③, we get

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx = \frac{\pi}{2}e^{-\lambda} \;\ldots\ldots ④$$

Using ④ in ①, we get

$$\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}.\frac{\pi}{2}\,e^{-\lambda} = \sqrt{\frac{\pi}{2}}\,e^{-\lambda}$$

Example 9 Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ and use it to evaluate $\int_0^\infty tan^{-1}\left(\frac{x}{a}\right) sinxdx$

Solution: To find Fourier sine transform

$$F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\lambda x\,dx$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin\lambda x\,dx$$

To evaluate the integral, differentiating both sides with respect to λ

$$\frac{d}{d\lambda}\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(\cos\lambda x)xdx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}cos\lambda xdx = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\lambda^2}$$

Now integrating both sides with respect to λ

$$\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int \frac{a}{a^2+\lambda^2}d\lambda$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{\lambda}{a}\right) + c$$

when $\lambda = 0$, $\bar{f}_s(\lambda) = 0$, $\Rightarrow$ $c = 0$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{\lambda}{a}\right)$$

Again taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty tan^{-1}\left(\frac{\lambda}{a}\right)\sin\lambda x\,d\lambda$$

Substituting $f(x) = \frac{e^{-ax}}{x}$ on both sides

$$\frac{e^{-ax}}{x} = \frac{2}{\pi}\int_0^\infty tan^{-1}\left(\frac{\lambda}{a}\right)\sin\lambda x\,d\lambda$$

Putting x =1 on both sides

$$\frac{\pi}{2}e^{-a} = \int_0^\infty tan^{-1}\frac{\lambda}{a}\sin\lambda\,d\lambda$$

$$\Rightarrow \int_0^\infty tan^{-1}\frac{\lambda}{a}\sin x\,dx = \frac{\pi}{2}e^{-a}$$

Example 10 If $t > 0$ Show that i. $\int_0^\infty \frac{\cos\lambda t}{\lambda^2+a^2}d\lambda = \frac{\pi}{2a}e^{-at}, a > 0$

$$\text{ii. } \int_0^\infty \frac{\lambda \sin \lambda t}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2} e^{at}, a \leq 0$$

Solution: i. Let $f(t) = \frac{\pi}{2a} e^{-at}, a > 0,\ t > 0$

Taking Fourier cosine transform of $f(t)$, we get

$$F_c\{f(t)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t \, dt$$

$$= \frac{\pi}{2a} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos \lambda t \, dt$$

$$= \frac{1}{a} \sqrt{\frac{\pi}{2}} \frac{a}{a^2+\lambda^2}$$

Also inverse Fourier cosine transform of $\bar{f}_c(\lambda)$ gives $f(t)$ as:

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda t \, d\lambda$$

$$= \frac{1}{a} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{a^2+\lambda^2} \cos \lambda t \, d\lambda$$

$$\Rightarrow f(t) = \int_0^\infty \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda$$

$$\therefore \int_0^\infty \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2a} e^{-at}, a > 0$$

ii. Again let $g(t) = \frac{\pi}{2} e^{at}, a \leq 0,\ t > 0$

Taking Fourier sine transform of $g(t)$, we get

$$F_s\{g(t)\} \equiv \bar{g}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \sin \lambda t \, dt$$

$$= \frac{\pi}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{at} \sin \lambda t \, dt, a \leq 0$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-at} \sin \lambda t \, dt\, , a > 0$$

$$= \sqrt{\frac{\pi}{2}} \frac{\lambda}{a^2+\lambda^2}$$

Also inverse Fourier sine transform of $\bar{g}_s(\lambda)$ gives $g(t)$ as:

$$g(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{g}_s(\lambda) \sin \lambda t \, d\lambda$$

$$= \sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\lambda}{a^2+\lambda^2}\sin\lambda t\,d\lambda$$

$$\Rightarrow g(t) = \int_0^\infty \frac{\lambda\sin\lambda t}{\lambda^2+a^2}d\lambda$$

$$\therefore \int_0^\infty \frac{\lambda\sin\lambda t}{\lambda^2+a^2}d\lambda = \frac{\pi}{2}e^{at}, a \le 0$$

Example 11 Prove that Fourier transform of $e^{\frac{-x^2}{2}}$ is self reciprocal.

Solution: Fourier transform of $f(x)$ is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}f(x)dx$$

$$\therefore F\left\{e^{\frac{-x^2}{2}}\right\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}e^{i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}+i\lambda x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2\,i\lambda x)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2i\lambda x+(i\lambda)^2-(i\lambda)^2)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-i\lambda)^2+\frac{i^2\lambda^2}{2}}\,dx$$

$$= \frac{e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-i\lambda)^2}\,dx$$

$$= \frac{e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}}dz \qquad \text{By putting } z = (x-i\lambda)$$

$$= \frac{2e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{\frac{-z^2}{2}}dz \qquad e^{\frac{-z^2}{2}} \text{ being even function of } z$$

$$= \frac{2e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-\left(\frac{z}{\sqrt{2}}\right)^2}dz$$

Put $\frac{z}{\sqrt{2}} = t \Rightarrow dz = \sqrt{2}\,dt$

$$\therefore \bar{f}(\lambda) = \frac{2\sqrt{2}e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-t^2}dt$$

$$= \frac{2e^{\frac{-\lambda^2}{2}}}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2} = e^{\frac{-\lambda^2}{2}} \qquad \because \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$$

$$\therefore F\left\{e^{\frac{-x^2}{2}}\right\} = e^{\frac{-\lambda^2}{2}}$$

Hence we see that Fourier transform of $e^{\frac{-x^2}{2}}$ is given by $e^{\frac{-\lambda^2}{2}}$. Variable x is transformed to λ. ∴ We can say that Fourier transform of $e^{\frac{-x^2}{2}}$ is self reciprocal.

Example 12 Find Fourier Cosine transform of e^{-x^2}.

Solution: By definition, $F_c\{f(x)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\, dx$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\left(\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{-x^2}e^{i\lambda x} + e^{-x^2}e^{-i\lambda x}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty (e^{-x^2+i\lambda x} + e^{-x^2-i\lambda x})\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{-\left(x^2-2\left(\frac{i\lambda}{2}\right)x+\left(\frac{i\lambda}{2}\right)^2-\left(\frac{i\lambda}{2}\right)^2\right)} + e^{-\left(x^2+2\left(\frac{i\lambda}{2}\right)x+\left(\frac{i\lambda}{2}\right)^2-\left(\frac{i\lambda}{2}\right)^2\right)}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{-\left(x-\frac{i\lambda}{2}\right)^2+\frac{i^2\lambda^2}{4}} + e^{-\left(x+\frac{i\lambda}{2}\right)^2+\frac{i^2\lambda^2}{4}}\right)dx$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}}\left[\int_0^\infty e^{-\left(x-\frac{i\lambda}{2}\right)^2}dx + \int_0^\infty e^{-\left(x+\frac{i\lambda}{2}\right)^2}dx\right]$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}}\left[\frac{\sqrt{\pi}}{2}+\frac{\sqrt{\pi}}{2}\right] = \frac{\sqrt{\pi}e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}}$$

$$\Rightarrow \bar{f_c}(\lambda) = \frac{1}{\sqrt{2}}e^{-\frac{\lambda^2}{4}}$$

Or

Fourier Cosine transform of e^{-x^2} can also be found using the method given below:

$$\bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos\lambda x\, dx \quad \text{....①}$$

Differentiating both sides with respect to λ

$$\Rightarrow \frac{d}{d\lambda}\bar{f_c}(\lambda) = -\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-x^2}\sin\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\sin \lambda x.\frac{e^{-x^2}}{2}\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty \lambda \cos \lambda x.\frac{e^{-x^2}}{2}\,dx$$

$$= 0 - \frac{\lambda}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos \lambda x\,dx \quad \ldots.②$$

$$\Rightarrow \frac{d}{d\lambda}\bar{f}_c(\lambda) = -\frac{\lambda}{2}\bar{f}_c(\lambda) \qquad \text{using ① in ②}$$

$$\Rightarrow \frac{\frac{d}{d\lambda}\bar{f}_c(\lambda)}{\bar{f}_c(\lambda)} = -\frac{\lambda}{2}$$

Integrating both sides with respect to λ

$$\Rightarrow \log \bar{f}_c(\lambda) = -\frac{\lambda^2}{4} + \log k$$, where $\log k$ is the constant of integration

$$\Rightarrow \bar{f}_c(\lambda) = K\mathrm{e}^{-\frac{\lambda^2}{4}} \ \ldots..③$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos \lambda x\,dx = k\mathrm{e}^{-\frac{\lambda^2}{4}}$$

Putting $\lambda = 0$ on both sides

$$\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}dx = k$$

$$\Rightarrow k = \sqrt{\frac{2}{\pi}}.\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} \ \ldots.. ④$$

Using ④ in ③, we get

$$\bar{f}_c(\lambda) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\lambda^2}{4}}$$

Example 13 Find Fourier transform of $xe^{-ax^2}, a > 0$

Solution: By definition, $F\{xe^{-ax^2}\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-ax^2}e^{i\lambda x}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-ax^2+i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-a\left(x^2-2\left(\frac{i\lambda}{2a}\right)x+\left(\frac{i\lambda}{2a}\right)^2-\left(\frac{i\lambda}{2a}\right)^2\right)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-a\left(x-\frac{i\lambda}{2a}\right)^2+\frac{i^2\lambda^2}{4a}}\,dx$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}\left(x-\frac{i\lambda}{2a}\right)e^{-a\left(x-\frac{i\lambda}{2a}\right)^2}\,dx + \frac{i\lambda}{2a}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{i\lambda}{2a}\right)^2}\,dx\right]$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} te^{-at^2}\ dt + \frac{i\lambda}{2a}\int_{-\infty}^{\infty} e^{-at^2}\ dt\right], \text{Putting } \left(x - \frac{i\lambda}{2a}\right) = t$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} \left[0 + \frac{i\lambda}{a}\int_{0}^{\infty} e^{-at^2}\ dt\right]$$

$\because te^{-at^2}$ is odd function and e^{-at^2} is even function in t

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} . \frac{i\lambda}{a}\int_{0}^{\infty} e^{-(\sqrt{a}\,t)^2}\ dt$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} . \frac{i\lambda}{a\sqrt{a}}\int_{0}^{\infty} e^{-z^2}\ dz \text{ , Putting } \sqrt{a}\,t = z$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} . \frac{i\lambda}{a\sqrt{a}} . \frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-z^2}\ dz = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \overline{f}(\lambda) = \frac{i\lambda\, e^{\frac{-\lambda^2}{4a}}}{2a\sqrt{2a}}$$

Example 14 Find Fourier cosine integral representation of $f(x) = \begin{cases} x^2, 0 < x < a \\ 0 \quad , \quad x > a \end{cases}$

Solution: Taking Fourier Cosine transform of $f(x) = \begin{cases} x^2, 0 < x < a \\ 0 \quad , \quad x > a \end{cases}$

$$F_c\{f(x)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\lambda x\, dx$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{a} x^2\cos\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(x^2)\left(\frac{sin\lambda x}{\lambda}\right) - (2x)\left(\frac{-cos\lambda x}{\lambda^2}\right) + (2)\left(\frac{-sin\lambda x}{\lambda^3}\right)\right]_0^a$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\left[\left(\frac{a^2}{\lambda} - \frac{2}{\lambda^3}\right)sin\lambda a + \frac{2a}{\lambda^2}cos\lambda a\right]$$

Now taking Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left[\left(\frac{a^2}{\lambda} - \frac{2}{\lambda^3}\right)sin\lambda a + \frac{2a}{\lambda^2}cos\lambda a\right]\cos\lambda x\, d\lambda$$

This is the required Fourier cosine integral representation of $f(x) = \begin{cases} x^2, 0 < x < a \\ 0 \quad , \quad x > a \end{cases}$

Example 15 If $f(x) = \begin{cases} \sin x\, , & 0 < x < \pi \\ 0, & otherwise \end{cases}$, prove that

$f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda$. Hence evaluate $\int_0^\infty \frac{\cos\frac{\pi t}{2}}{1-t^2}\,dt$

Solution: Given $f(x) = \begin{cases}\sin x, & 0 < x < \pi \\ 0, & otherwise\end{cases}$

To find Fourier cosine integral representation of $f(x)$, taking Fourier Cosine transform of $f(x)$

$$\bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) cos\lambda x dx = \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x\, cos\lambda x dx$$

$$= \sqrt{\frac{1}{2\pi}}\int_0^\pi (\sin(\lambda+1)x - \sin(\lambda-1)x)\,dx$$

$$= \sqrt{\frac{1}{2\pi}}\left[-\frac{\cos(\lambda+1)x}{(\lambda+1)} + \frac{\cos(\lambda-1)x}{(\lambda-1)}\right]_0^\pi$$

$$= \sqrt{\frac{1}{2\pi}}\left[-\frac{\cos(\lambda+1)\pi}{(\lambda+1)} + \frac{\cos(\lambda-1)\pi}{(\lambda-1)} + \frac{1}{(\lambda+1)} - \frac{1}{(\lambda-1)}\right]$$

$$= \sqrt{\frac{1}{2\pi}}\left[\frac{\cos\lambda\pi}{(\lambda+1)} - \frac{\cos\lambda\pi}{(\lambda-1)} + \frac{1}{(\lambda+1)} - \frac{1}{(\lambda-1)}\right]$$

$$= \sqrt{\frac{1}{2\pi}}\left[\frac{(\lambda-1)\cos\lambda\pi - (\lambda+1)\cos\lambda\pi}{(\lambda+1)(\lambda-1)} + \frac{\lambda-1-\lambda-1}{(\lambda+1)(\lambda-1)}\right]$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{1}{2\pi}}\left[\frac{-2\cos\lambda\pi - 2}{\lambda^2-1}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1+\cos\lambda\pi}{1-\lambda^2}\right]$$

Taking Inverse Fourier Cosine transform, $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f_c}(\lambda)\cos\lambda x\,d\lambda$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{1+\cos\lambda\pi}{1-\lambda^2}\right]\cos\lambda x\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + 2\cos\lambda\pi\cos\lambda x}{1-\lambda^2}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda$$

Putting $f(x) = \begin{cases}\sin x, & 0 < x < \pi \\ 0, & otherwise\end{cases}$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda = \begin{cases}\sin x, & 0 < x < \pi \\ 0, & otherwise\end{cases}$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \frac{2\cos\frac{\pi\lambda}{2} + \cos\left(\pi+\frac{\pi}{2}\right)\lambda+\cos\left(\pi-\frac{\pi}{2}\right)\lambda}{1-\lambda^2}d\lambda = 1$$

$$\Rightarrow \int_0^\infty \frac{\cos\frac{\pi\lambda}{2}}{1-\lambda^2}d\lambda = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\cos\frac{\pi t}{2}}{1-t^2}dt = \frac{\pi}{2}$$

Example 16 Solve the integral equation $\int_0^\infty f(x)cos\lambda x dx = \begin{cases} 1-\lambda, 0 \le \lambda \le 1 \\ 0, \quad \lambda > 1 \end{cases}$

Hence deduce that $\int_0^\infty \frac{sin^2 t}{t^2}dt = \frac{\pi}{2}$

Solution: Given that $\int_0^\infty f(x)cos\lambda x dx = \begin{cases} 1-\lambda, 0 \le \lambda \le 1 \\ 0, \quad \lambda > 1 \end{cases}$ ……①

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cos\lambda x dx = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), 0 \le \lambda \le 1 \\ 0, \ \lambda > 1. \end{cases}$$

$$\Rightarrow \bar{f_c}(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), 0 \le \lambda \le 1 \\ 0, \ \lambda > 1. \end{cases}$$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f_c}(\lambda)\cos\lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^1 (1-\lambda)\cos\lambda x \ d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{sin\lambda x}{x}\right) - (-1)\left(\frac{-cos\lambda x}{x^2}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[-\frac{cosx}{x^2} + \frac{1}{x^2}\right] = \frac{2}{\pi}\left[\frac{1-cosx}{x^2}\right] = \frac{2}{\pi}\frac{2sin^2\frac{x}{2}}{x^2}$$

$$\Rightarrow f(x) = \frac{4sin^2\frac{x}{2}}{\pi x^2} \text{ ....②}$$

Using ②in ①, we get

$$\int_0^\infty \frac{4sin^2\frac{x}{2}}{\pi x^2} cos\lambda x dx = \begin{cases} 1-\lambda, 0 \le \lambda \le 1 \\ 0, \quad \lambda > 1 \end{cases}$$

Putting $\lambda = 0$ on both sides

$$\Rightarrow \frac{4}{\pi}\int_0^\infty \frac{sin^2\frac{x}{2}}{x^2}dx = 1$$

$$\Rightarrow \int_0^\infty \frac{sin^2\frac{x}{2}}{x^2}dx = \frac{\pi}{4}$$

Putting $\frac{x}{2} = t\,,\; dx = 2dt$

$$\therefore \int_0^\infty \frac{sin^2 t}{4t^2} 2dt = \frac{\pi}{4} \;\Rightarrow \int_0^\infty \frac{sin^2 t}{t^2} dt = \frac{\pi}{2}$$

Example 17 Find the function $f(x)$ if its Cosine transform is given by:

(i) $\frac{\sin a\lambda}{\lambda}$ (ii) $\begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\lambda}{2}\right), & \lambda < 2a \\ 0 \quad , & \lambda \geq 2a \end{cases}$

Solution: (i) Given that $\bar{f_c}(\lambda) = \frac{\sin a\lambda}{\lambda}$

Taking Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f_c}(\lambda) \cos \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda} \cos \lambda x \, d\lambda$$

$$= \frac{1}{2}.\frac{2}{\pi}\int_0^\infty \frac{2\sin a\lambda \cos \lambda x}{\lambda} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)\lambda}{\lambda} d\lambda + \frac{1}{\pi}\int_0^\infty \frac{\sin\,(a-x)\,\lambda}{\lambda} d\lambda$$

Now $0 < x < \infty \quad \therefore \; a + x > 0$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right], & a - x > 0 \;\; i.e. x < a \\ \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right], & a - x < 0 \;\; i.e. x > a \end{cases} \qquad \because \int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}, \lambda > 0$$

$$\Rightarrow f(x) = \begin{cases} 1 & , x < a \\ 0, & x > a \end{cases}$$

(ii) Given that $\bar{f_c}(\lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\lambda}{2}\right), & \lambda < 2a \\ 0 \quad , & \lambda \geq 2a \end{cases}$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f_c}(\lambda) \cos \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^{2a} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\lambda}{2}\right) \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi}\int_0^{2a} \left(a - \frac{\lambda}{2}\right) \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi}\left[\left(a - \frac{\lambda}{2}\right)\left(\frac{\sin \lambda x}{x}\right) - \left(-\frac{1}{2}\right)\left(-\frac{\cos \lambda x}{x^2}\right)\right]_0^{2a}$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2ax}{2x^2} + \frac{1}{2x^2}\right] = \frac{1}{2\pi x^2}[1 - \cos 2ax] = \frac{\sin^2 ax}{\pi x^2}$$

Example 18 Find the function $f(x)$ if its Sine transform is given by:

(i) $e^{-a\lambda}$ (ii) $\frac{\lambda}{1+\lambda^2}$

Solution: (i) Given that $\bar{f_s}(\lambda) = e^{-a\lambda}$

Taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f_s}(\lambda)\sin\lambda x\, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty e^{-a\lambda}\sin\lambda x\, d\lambda = \frac{2}{\pi}.\frac{x}{a^2+x^2}$$

(ii) Given that $\bar{f_s}(\lambda) = \frac{\lambda}{1+\lambda^2}$

Taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f_s}(\lambda)\sin\lambda x\, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2}\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda^2}{\lambda(1+\lambda^2)}\sin\lambda x\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{(1+\lambda^2)-1}{\lambda(1+\lambda^2)}\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda x}{\lambda}d\lambda - \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda x}{\lambda(1+\lambda^2)}d\lambda$$

$$\Rightarrow f(x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda x}{\lambda(1+\lambda^2)}d\lambda \quad \ldots\ldots①$$

$$\because \int_0^\infty \frac{\sin\lambda x}{\lambda}d\lambda = \frac{\pi}{2}, x > 0$$

Differentiating with respect to x

$$\Rightarrow f'(x) = 0 - \frac{2}{\pi}\int_0^\infty \frac{\lambda\cos\lambda x}{\lambda(1+\lambda^2)}d\lambda$$

$$\Rightarrow f'(x) = -\frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{(1+\lambda^2)}d\lambda \ \ldots\ldots②$$

Also $f''(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda\ \sin\lambda x}{(1+\lambda^2)}d\lambda = f(x)$

$$\Rightarrow f''(x) - f(x) = 0\ldots..\ ③$$

This is a linear differential equation with constant coefficients

③may be written as $(D^2 - 1)f(x) = 0$

Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow \mathrm{m} = \pm 1$

Solution of ③ is given by

$f(x) = c_1 e^{x} + c_2 e^{-x}$ ….. ④

$\Rightarrow f'(x) = c_1 e^{x} - c_2 e^{-x}$ …..⑤

Now from ①, $f(x) = 1$, at $x = 0$

Using in ④, we get $c_1 + c_2 = 1$ ….⑥

Again from ②, $f'(x) = -\frac{2}{\pi}\int_0^\infty \frac{1}{(1+\lambda^2)} d\lambda$, at $x = 0$

$\Rightarrow f'(x) = -\frac{2}{\pi}[\tan^{-1}\lambda]_0^\infty = -1$ at $x = 0$

Using in ⑤, we get $c_1 - c_2 = -1$ …. ⑦

Solving ⑥ and ⑦, we get $c_1 = 0,\ c_2 = 1$

Using in ④, we get $f(x) = e^{-x}$

Note: Solution of the differential equation $f''(x) - f(x) = 0$ may be written directly as $f(x) = e^{-x}$

Example 19 Find the Fourier transform of the function $f(x) = e^{-a|x|},\ -\infty < x < \infty$

Solution: $f(x) = \begin{cases} e^{ax}, x < 0 \\ e^{-ax} x \geq 0 \end{cases}$

Fourier transform of $f(x)$ is given by $F\{f(x)\} \equiv \bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx$

$$\Rightarrow \bar{f}(\lambda) = \int_{-\infty}^{0} e^{ax} e^{i\lambda x}\, dx + \int_0^\infty e^{-ax} e^{i\lambda x}\, dx$$

$$= \int_{-\infty}^{0} e^{x(a+i\lambda)} dx + \int_0^\infty e^{-x(a-i\lambda)} dx$$

$$= \left[\frac{e^{x(a+i\lambda)}}{(a+i\lambda)}\right]_{-\infty}^{0} - \left[\frac{e^{-x(a-i\lambda)}}{(a-i\lambda)}\right]_0^\infty$$

$$\Rightarrow \bar{f}(\lambda) = \frac{1}{a+i\lambda} + \frac{1}{a-i\lambda} = \frac{2a}{a^2+\lambda^2}$$

$$\therefore F\left\{e^{-a|x|}\right\} = \frac{2a}{a^2+\lambda^2}$$

Result:

$$\boldsymbol{F\left\{e^{-a|x|}\right\} = \frac{2a}{a^2+\lambda^2} \Rightarrow F^{-1}\left[\frac{2a}{a^2+\lambda^2}\right] = e^{-a|x|}}$$

Example 20 Find $F^{-1}\left[\frac{1}{(9+\lambda^2)(4+\lambda^2)}\right]$

Solution: $F^{-1}\left[\frac{1}{(9+\lambda^2)(4+\lambda^2)}\right] = \frac{1}{5}F^{-1}\left[-\frac{1}{9+\lambda^2}+\frac{1}{4+\lambda^2}\right]$

$$= \frac{1}{5}F^{-1}\left[-\frac{1}{3^2+\lambda^2}+\frac{1}{2^2+\lambda^2}\right]$$

$$= \frac{-1}{30}F^{-1}\left[\frac{6}{9+\lambda^2}\right]+\frac{1}{20}F^{-1}\left[\frac{4}{4+\lambda^2}\right]$$

$$= \frac{-1}{30}e^{-3|x|}+\frac{1}{20}e^{-2|x|} \qquad \because F^{-1}\left[\frac{2a}{a^2+\lambda^2}\right]=e^{-a|x|}$$

Example 21 Find the Fourier transform of the function $f(x)=e^{-ax}U(x),\ a>0$ where $U(x)$ represents unit step function

Solution: $f(x)=e^{-ax}\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases} = \begin{cases}0, & x<0\\ e^{-ax}, & x\geq 0\end{cases}$

Fourier transform of $f(x)$ is given by $F\{f(x)\}\equiv \bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}f(x)dx$

$$\Rightarrow \bar{f}(\lambda) = \int_0^{\infty} e^{-ax}e^{i\lambda x}\,dx$$

$$= \int_0^{\infty} e^{-x(a-i\lambda)}dx$$

$$= -\left[\frac{e^{-x(a-i\lambda)}}{(a-i\lambda)}\right]_0^{\infty}$$

$$\Rightarrow \bar{f}(\lambda) = \frac{1}{a-i\lambda}$$

$$\therefore F\{f(x)\} = \frac{1}{a-i\lambda}$$

or $F\{e^{-ax}U(x)\} = \frac{1}{a-i\lambda}$

Result:

$$\boldsymbol{F\{e^{-ax}U(x)\} = \frac{1}{a-i\lambda} \Rightarrow F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}U(x) = e^{-ax}H(x)}$$

Note: If Fourier transform of $f(x)=e^{-ax}U(x)$ is taken as $\int_{-\infty}^{\infty} e^{-i\lambda x}e^{-ax}U(x)dx$, then $F^{-1}\left[\frac{1}{a+i\lambda}\right] = e^{-ax}U(x) = e^{-ax}H(x)$

Example 22 Find the inverse transform of the following functions:

i. $\frac{1}{2-3i\lambda-\lambda^2}$ ii. $\frac{1}{8+6i\lambda-\lambda^2}$ iii. $\frac{5}{6-5i\lambda-\lambda^2}$

Solution: i. $F^{-1}\left[\frac{1}{2-3i\lambda-\lambda^2}\right] = F^{-1}\left[\frac{1}{(1-i\lambda)(2-i\lambda)}\right] = F^{-1}\left[\frac{1}{(1-i\lambda)}-\frac{1}{(2-i\lambda)}\right]$

$$= F^{-1}\left[\frac{1}{(1-i\lambda)}\right] - F^{-1}\left[\frac{1}{(2-i\lambda)}\right]$$

$$= e^{-x}H(x) - e^{-2x}H(x) \qquad \because F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$$

$$\Rightarrow F^{-1}\left[\frac{1}{2-3i\lambda-\lambda^2}\right] = \begin{cases}(e^{-x}-e^{-2x}), x \geq 0\\ \quad 0\ , \quad x < 0\end{cases}$$

ii. $F^{-1}\left[\frac{1}{8+6i\lambda-\lambda^2}\right] = F^{-1}\left[\frac{1}{(4+i\lambda)(2+i\lambda)}\right] = F^{-1}\left[\frac{1}{(4+i\lambda)} - \frac{1}{(2+i\lambda)}\right]$

$$= F^{-1}\left[\frac{1}{(4+i\lambda)}\right] - F^{-1}\left[\frac{1}{(2+i\lambda)}\right]$$

$$= e^{-4x}H(x) - e^{-2x}H(x) \qquad \because F^{-1}\left[\frac{1}{a+i\lambda}\right] = e^{-ax}H(x)$$

$$\Rightarrow F^{-1}\left[\frac{1}{8+6i\lambda-\lambda^2}\right] = \begin{cases}(e^{-x}-e^{-2x}), x \geq 0\\ \quad 0\ , \quad x < 0\end{cases}$$

iii. $F^{-1}\left[\frac{5}{6-5i\lambda-\lambda^2}\right] = 5\,F^{-1}\left[\frac{1}{(2-i\lambda)(3-i\lambda)}\right] = 5F^{-1}\left[\frac{1}{(2-i\lambda)} - \frac{1}{(3-i\lambda)}\right]$

$$= 5F^{-1}\left[\frac{1}{(2-i\lambda)}\right] - 5F^{-1}\left[\frac{1}{(3-i\lambda)}\right]$$

$$= 5e^{-2x}H(x) - 5e^{-3x}H(x) \qquad \because F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$$

$$\Rightarrow F^{-1}\left[\frac{5}{6-5i\lambda-\lambda^2}\right] = \begin{cases}5(e^{-2x}-e^{-3x}), x \geq 0\\ \quad 0\ , \quad x < 0\end{cases}$$

Example 23 Find the Fourier transform of $f(x) = \frac{1}{2-ix}$

Solution: We know $F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$

$$\Rightarrow F^{-1}\left[\frac{1}{2-i\lambda}\right] = e^{-2x}\mathrm{H}(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2-i\lambda}e^{-i\lambda x}\,d\lambda = e^{-2x}\mathrm{H}(x)$$

Interchanging x and λ, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2-ix}e^{-i\lambda x}\,dx = e^{-2\lambda}\mathrm{H}(\lambda)$$

$$= \begin{cases}0\ , & \lambda < 0\\ e^{-2\lambda}, & \lambda \geq 0\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{2-ix}e^{-i\lambda x}\,dx = \begin{cases}0\ , & \lambda < 0\\ 2\pi e^{-2\lambda}, & \lambda \geq 0\end{cases}$$

$$\Rightarrow \mathrm{F}\left\{\frac{1}{2-ix}\right\} = \begin{cases} 0\,, & \lambda < 0 \\ 2\pi e^{-2\lambda}, & \lambda \geq 0 \end{cases}$$

Properties of Fourier Transforms

Linearity: If $\overline{f}(\lambda)$ and $\overline{g}(\lambda)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F\{af(x) + bg(x)\} = a\overline{f}(\lambda) + b\overline{g}(\lambda)$$

Proof: $F\{af(x) + bg(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]\, e^{i\lambda x} dx$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{i\lambda x} dx$$

$$= a\overline{f}(\lambda) + b\overline{g}(\lambda)$$

Change of scale: If $\overline{f}(\lambda)$ is Fourier transforms of $f(x)$, then $F\{f(ax)\} = \frac{1}{a}\overline{f}\left(\frac{\lambda}{a}\right)$

Proof: $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax).\, e^{i\lambda x}$

Putting $ax = t \Rightarrow adx = dt$

$$\therefore F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\lambda \frac{t}{a}}.\, \frac{dt}{a} = \frac{1}{a}.\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\left(\frac{\lambda}{a}\right)t} dt$$

$$= \frac{1}{a}\overline{f}\left(\frac{\lambda}{a}\right)$$

Shifting Property: If $\overline{f}(\lambda)$ is Fourier transforms of $f(x)$, then $F\{f(x-a)\} = e^{i\lambda a}\, \overline{f}(\lambda)$

Proof: $F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a).\, e^{i\lambda x}$

Putting $(x-a) = t \Rightarrow dx = dt$

$$\therefore F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\lambda(t+a)}\, dt$$

$$= e^{i\lambda a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\lambda t} dt = e^{i\lambda a}\, \overline{f}(\lambda)$$

Modulation Theorem: If $\overline{f}(\lambda)$ is Fourier transforms of $f(x)$, then

i. $F\{f(x)\cos ax\} = \frac{1}{2}\left\{\overline{f}(\lambda + a) + \overline{f}(\lambda - a)\right\}$

ii. $F_s[f(x)\cos ax] = \frac{1}{2}\left\{\overline{f}_s(\lambda + a) + \overline{f_s}(\lambda - a)\right\}$

iii. $F_c[f(x)\sin ax] = \frac{1}{2}\left\{\overline{f}_s(\lambda + a) - \overline{f_s}(\lambda - a)\right\}$

iv. $F_c[f(x)\cos ax] = \frac{1}{2}\left\{\overline{f}_c(\lambda + a) + \overline{f_c}(\lambda - a)\right\}$

v. $F_s[f(x)\sin ax] = \frac{1}{2}\{\overline{f}_c(\lambda - a) - \overline{f_c}(\lambda + a)\}$

Proof: i. $F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax \,.\, e^{i\lambda x}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{e^{iax}+e^{-iax}}{2}e^{i\lambda x}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(\lambda+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(\lambda-a)x}dx\right]$$

$$= \frac{1}{2}\{\overline{f}(\lambda + a) + \overline{f}(\lambda - a)\}$$

ii. $F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\ \sin\lambda x\ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\sin(\lambda + a)x + \sin(\lambda - a)x\ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda + a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda - a)x\,dx\right]$$

$$= \frac{1}{2}\{\overline{f}_s(\lambda + a) + \overline{f_s}(\lambda - a)\}$$

iii. $F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax\ \cos\lambda x\ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\sin(\lambda + a)x - \sin(\lambda - a)x\ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda + a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda - a)x\,dx\right]$$

$$= \frac{1}{2}\{\overline{f}_s(\lambda + a) - \overline{f_s}(\lambda - a)\}$$

iv. $F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\ \cos\lambda x\ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\cos(\lambda + a)x + \cos(\lambda - a)x\ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\lambda + a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\lambda - a)x\,dx\right]$$

$$= \frac{1}{2}\{\overline{f}_c(\lambda + a) + \overline{f_c}(\lambda - a)\}$$

v. $F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax\ \sin\lambda x\ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,[\cos(\lambda-a)x - \cos(\lambda+a)x\ \ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\lambda-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\lambda+a)x\,dx\right]$$

$$= \frac{1}{2}\left\{\overline{f}_c(\lambda-a) - \overline{f_c}(\lambda+a)\right\}$$

Convolution theorem: Convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) *\ g(x) =\ \int_{-\infty}^{\infty} f(u)g(x-u)du$$

If $\overline{f}(\lambda)$ and $\overline{g}(\lambda)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then Convolution theorem for Fourier transforms states that

$$F\{f(x) * g(x)\} =\ F\{f(x)\}\,.\,F\{fg(x)\} \equiv \overline{f}(\lambda).\,\overline{g}(\lambda)$$

Proof: By definition $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx$ and $\bar{g}(\lambda) =\ \int_{-\infty}^{\infty} e^{i\lambda x} g(x)dx$

Now $\quad f(x) *\ g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$

$$\therefore F\{f(x) * g(x)\} =\ \int_{-\infty}^{\infty} e^{i\lambda x}\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]dx$$

Changing the order of integration, we get

$$\therefore F\{f * g\} =\ \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{i\lambda x} g(x-u)dx\right]du$$

Putting $x - u = t \ \ \Rightarrow dx = dt$ in the inner integral, we get

$$F\{f * g\} =\ \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{i\lambda(u+t)} g(t)dt\right]du$$

$$=\ \int_{-\infty}^{\infty} e^{i\lambda u} f(u)\left[\int_{-\infty}^{\infty} e^{i\lambda t} g(t)dt\right]du$$

$$= \int_{-\infty}^{\infty} e^{i\lambda u} f(u)\overline{g}(\lambda)\ du$$

$$= \overline{g}(\lambda)\int_{-\infty}^{\infty} e^{i\lambda u} f(u)du$$

$$= \overline{f}(\lambda).\,\overline{g}(\lambda)$$

Example 24 Find the Fourier transform of e^{-x^2}. Hence find Fourier transforms of

i. $e^{-ax^2}, a > 0$ ii. $e^{\frac{-x^2}{2}}$ iii. $e^{\,2(x-3)^2}$ iv. $e^{-x^2}\cos 2x$

Solution: $F\left\{e^{-x^2}\right\} = \overline{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{i\lambda x}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2+i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x^2 - 2\left(\frac{i\lambda}{2}\right)x + \left(\frac{i\lambda}{2}\right)^2 - \left(\frac{i\lambda}{2}\right)^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{i\lambda}{2}\right)^2 + \frac{i^2\lambda^2}{4}} dx$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{i\lambda}{2}\right)^2} \, dx$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \qquad \text{By putting } z = \left(x - \frac{i\lambda}{2}\right)$$

$$= \frac{2e^{-\frac{\lambda^2}{4}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2} dz \qquad e^{-z^2} \text{ being even function of } z$$

$$\therefore \overline{f}(\lambda) = \frac{2e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} . \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} e^{\frac{-\lambda^2}{4}} \text{ .....①}$$

∴ We have $F\{f(x)\} = \overline{f}(\lambda) = \frac{1}{\sqrt{2}} e^{\frac{-\lambda^2}{4}}$ if $f(x) = e^{-x^2}$

i. Now $F\{e^{-ax^2}\} = F\left\{e^{(\sqrt{a}\,\sqrt{x})^2}\right\}$

$$= \frac{1}{\sqrt{a}} \overline{f}\left(\frac{\lambda}{\sqrt{a}}\right) \qquad \text{By change of scale property.... ②}$$

$$\therefore F\{e^{-ax^2}\} = \frac{1}{\sqrt{a}} . \frac{1}{\sqrt{2}} e^{-\frac{1}{4}\left(\frac{\lambda}{\sqrt{a}}\right)^2} = \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}} \qquad \text{Using ①in ②}$$

ii. Putting $a = \frac{1}{2}$ in i.

$$F\left\{e^{-\frac{x^2}{2}}\right\} = \frac{1}{\sqrt{2.\frac{1}{2}}} . e^{-\frac{\lambda^2 .}{\frac{4}{.2}}} = e^{-\frac{\lambda^2}{2}}$$

iii. To find $F\{e^{-2(x-3)^3}\}$, Put $a = 2$ in i.

$$F\{e^{-2x^2}\} = \frac{1}{2} e^{-\frac{\lambda^2}{8}}$$

$$\therefore F\{e^{-2(x-3)^2}\} = e^{3i\lambda} . \frac{1}{2} e^{-\frac{\lambda^2}{8}} \quad \because \text{By shifting property } F\{f(x-k)\} = e^{i\lambda k}\overline{f}(\lambda)$$

iv. To find Fourier transform of $F\{ e^{-x^2} cos2x\}$

$$F\{f(x)\cos ax\} = \frac{1}{2} \overline{f}(\lambda + a) + \overline{f}(\lambda - a) \quad \text{By modulation theorem}$$

Now $F\{e^{-x^2}\} \equiv \overline{f}(\lambda) = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}}$.

$$\therefore\ F\{e^{-x^2}cos2x\} = \frac{1}{2}\left[\frac{1}{\sqrt{2}}e^{-\frac{(\lambda+2)^2}{4}} + \frac{1}{\sqrt{2}}e^{-\frac{(\lambda-2)^2}{4}}\right]$$

Example 25 Using Convolution theorem, find $F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right]$

Solution: $F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right] = F^{-1}\left[\frac{1}{(4-i\lambda)(3-i\lambda)}\right] = F^{-1}\left[\frac{1}{(4-i\lambda)}.\frac{1}{(3-i\lambda)}\right]$

Now by Convolution theorem

$$F\{f(x) * g(x)\} = \bar{f}(\lambda).\bar{g}(\lambda) \Rightarrow F^{-1}[\bar{f}(\lambda).\bar{g}(\lambda)] = f(x) * g(x)$$

$$\therefore F^{-1}\left[\frac{1}{(4-i\lambda)}.\frac{1}{(3-i\lambda)}\right] = F^{-1}\left[\frac{1}{(4-i\lambda)}\right] * F^{-1}\left[\frac{1}{(3-i\lambda)}\right]$$

$$= e^{-4x}H(x) * e^{-3x}H(x) \qquad \because F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$$

$$= \int_{-\infty}^{\infty} e^{-4u}H(u)e^{-3(x-u)}H(x-u)du$$

$$\because f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

$$= e^{-3x}\int_{-\infty}^{\infty} e^{-u}H(u)H(x-u)du$$

Now $H(u)H(x-u) = \begin{cases} 1, & u \geq 0, x-u \geq 0, \quad i.e.\ 0 \leq u \leq x \\ 0, u < 0, x-u < 0, & i.e.\ u < 0 \text{ and } u > x \end{cases}$

$$\therefore F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right] = e^{-3x}\int_0^x e^{-u}du = -e^{-3x}[e^{-u}]_0^x = -e^{-3x}[e^{-x}-1], x \geq 0$$

$$= e^{-3x} - e^{-4x}, x \geq 0$$

$$\Rightarrow F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right] = \begin{cases} e^{-3x} - e^{-4x}, x \geq 0 \\ 0, \quad x < 0 \end{cases}$$

Example 26 Find the inverse Fourier transforms of $\frac{e^{3i\lambda}}{2-i\lambda}$

Solution: i. We know that $F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$

$$\therefore F^{-1}\left[\frac{1}{2-i\lambda}\right] = e^{-2x}H(x)$$

Now By shifting property $F\{f(x-k)\} = e^{i\lambda k}\bar{f}(\lambda)$

$$\Rightarrow F^{-1}[e^{i\lambda k}\bar{f}(\lambda)] = f(x-k)$$

$$\therefore F^{-1}\left[\frac{e^{3i\lambda}}{2-i\lambda}\right] = e^{-2(x-3)}H(x-3)$$

Fourier Transforms of Derivatives

Let $u(x,t)$ be a function of two independent variables x and t, such that Fourier transform of $u(x,t)$ is denoted by $\overline{u}(\lambda,t)$ i.e $\overline{u}(\lambda,t) = \int_{-\infty}^{\infty} e^{i\lambda x} u(x,t)dx$

Again let $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots \to 0$ as $x \to \pm\infty$,

Then Fourier transforms of $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots$ with respect to x are given by:

1. $F\left\{\frac{\partial u}{\partial x}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial u}{\partial x}dx = \left[e^{i\lambda x}u\right]_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}udx = -i\lambda\,\overline{u}(\lambda,t)$

   $F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial^2 u}{\partial x^2}dx = \left[e^{i\lambda x}\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial u}{\partial x}dx = (-i\lambda)^2\,\overline{u}(\lambda,t)$

   ⋮

   $\boldsymbol{F\left\{\frac{\partial^n u}{\partial x^n}\right\} = (-i\lambda)^n\,\overline{u}(\lambda,t)}$

2. Fourier sine transform of $\frac{\partial^2 u}{\partial x^2}$ is given by:

   $F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty}\frac{\partial^2 u}{\partial x^2}\sin\lambda x\,dx = \left[\sin\lambda x\frac{\partial u}{\partial x}\right]_0^{\infty} - \lambda\int_0^{\infty}\cos\lambda x\frac{\partial u}{\partial x}dx$

   $= 0 - \lambda[\cos\lambda x\,.\,u(x,t)]_0^{\infty} - \lambda^2\int_0^{\infty}\sin\lambda x\frac{\partial^2 u}{\partial x^2}dx$

   $\therefore \boldsymbol{F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \lambda u(0,t) - \lambda^2\overline{u}_s(\lambda,t)}$

3. Fourier cosine transform of $\frac{\partial^2 u}{\partial x^2}$ is given by:

   $F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty}\frac{\partial^2 u}{\partial x^2}\cos\lambda x\,dx = \left[\cos\lambda x\frac{\partial u}{\partial x}\right]_0^{\infty} + \lambda\int_0^{\infty}\sin\lambda x\frac{\partial u}{\partial x}dx$

   $= -\left[\frac{\partial u}{\partial x}\right]_{x=0} + \lambda[\sin\lambda x\,.\,u(x,t)]_0^{\infty} - \lambda^2\int_0^{\infty}\cos\lambda x\frac{\partial^2 u}{\partial x^2}dx$

   $\therefore \boldsymbol{F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - \lambda^2\overline{u}_c(\lambda,t)}$

4. Fourier transforms of $\frac{\partial u}{\partial t}$ with respect to x are given by:

   $F\left\{\frac{\partial u}{\partial t}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial u}{\partial t}dx = \frac{d}{dt}\int_{-\infty}^{\infty} e^{i\lambda x}u(x,t)dx$

   $\therefore \boldsymbol{F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}(\lambda,t)}$

   Similarly $\boldsymbol{F_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_s(\lambda,t)}$

   $\boldsymbol{F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_c(\lambda,t)}$

2.5 Applications of Fourier Transforms to boundary value problems

Partial differential equation together with boundary and initial conditions can be easily solved using Fourier transforms. In one dimensional boundary value problems, the partial differential equations can easily be transformed into an ordinary differential equation by applying a suitable transform and solution to boundary value problem is obtained by applying inverse transform. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

Algorithm to solve partial differential equations with boundary values:

1. Apply the suitable transform to given partial differential equation. For this check the range of x
   i. If $-\infty < x < \infty$, then apply Fourier transform.
   ii. If $0 < x < \infty$, then check initial value conditions
      a) If value of $u(0,t)$ is given, then apply Fourier sine transform
      b) If value of $\left[\frac{\partial u}{\partial x}\right]_{x=0}$ is given, then apply Fourier cosine transform

   An ordinary differential equation will be formed after applying the transform.
2. Solve the differential equation using usual methods.
3. Apply Boundary value conditions to evaluate arbitrary constants.
4. Apply inverse transform to get the required expression for $u(x,t)$.

Example 27 The temperature $u(x,t)$ at any point of an infinite bar satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty,\ t > 0$ and the initial temperature along the length of the bar is given by $u(x,0) = \begin{cases} 1 \; for \; |x| < 1 \\ 0 \; for \; |x| > 1 \end{cases}$

Determine the expression for $u(x,t)$.

Solution: As range of x is $(-\infty, \infty)$, applying Fourier transform to both sides of the given equation :

$$F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt}\overline{u}(\lambda,t) = -\lambda^2\overline{u}(\lambda,t) \quad \because F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}(\lambda,t) \text{ and } F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = (-i\lambda)^2\,\overline{u}(\lambda,t)$$

Rearranging the ordinary differential equation in variable separable form:

$$\Rightarrow \frac{d\overline{u}}{\overline{u}} = -\lambda^2 dt \ldots ① \qquad \text{where } \overline{u} \approx \overline{u}(\lambda,t)$$

Solving ① using usual methods of variable separable differential equations

$$\log \overline{u} = -\lambda^2 t + \log A$$

$\Rightarrow \log\frac{\overline{u}}{A} = -\lambda^2 t$

$\Rightarrow \overline{u}(\lambda, t) = A\, e^{-\lambda^2 t}$ ... ②

Putting $t = 0$ on both sides

$\Rightarrow \overline{u}(\lambda, 0) = A$ ... ③

Now given that $u(x, 0) = \begin{cases} 1 \; for \; |x| < 1 \\ 0 \; for \; |x| > 1 \end{cases}$

Taking Fourier transform on both sides, we get

$\Rightarrow \overline{u}\,(\lambda, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0)\, e^{\,i\lambda x} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{\,i\lambda x}\, dx$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{i\lambda} \left[e^{\,i\lambda x}\right]_{-1}^{1}$

$= \frac{1}{\sqrt{2\pi}} \frac{1}{i\lambda} \left[e^{\,i\lambda} - e^{-i\lambda}\right] = \frac{1}{\sqrt{2\pi}} \frac{2i}{i\lambda} \left[\frac{e^{\,i\lambda} - e^{-i\lambda}}{2i}\right]$

$\Rightarrow \; \overline{u}\,(\lambda, 0) = \frac{2}{\sqrt{2\pi}} \frac{sin\lambda}{\lambda}$ ... ④

From ③and ④, we get

$A = \frac{2}{\sqrt{2\pi}} \frac{sin\lambda}{\lambda}$ ... ⑤

Using ⑤ in ②, we get

$\overline{u}(\lambda, t) = \frac{2}{\sqrt{2\pi}} \frac{sin\lambda}{\lambda}\; e^{-\lambda^2 t}$

Taking Inverse Fourier transform

$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} \overline{u}(\lambda, t) d\lambda$

$\Rightarrow u(x, t) = \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{sin\lambda}{\lambda}\; e^{-\lambda^2 t} e^{-i\lambda x} d\lambda$

$\Rightarrow u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{sin\lambda}{\lambda} e^{-\lambda^2 t} (\cos\lambda x - i \sin\lambda x) d\lambda$

$\Rightarrow u(x, t) = \frac{2}{\pi} \int_{0}^{\infty} e^{-\lambda^2 t} \left(\frac{\sin\lambda \cos\lambda x}{\lambda}\right) d\lambda \qquad \because \left(\frac{\sin\lambda \sin\lambda x}{\lambda}\right)$ is odd function of λ

Example 28 Using Fourier transform, solve the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, 0 < x < \infty,\; t > 0$ subject to conditions:

i. $u(0,t) = 0, \quad t > 0$

ii. $u(x,0) = e^{-x}, \;\; x > 0$

iii. u and $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \pm\infty$

Solution: As range of x is $(0,\infty)$, and also value of $u(0,t)$ is given in initial value conditions, applying Fourier sine transform to both sides of the given equation:

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = kF_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt}\overline{u}_s(\lambda,t) = k\lambda u(0,t) - k\lambda^2\overline{u}_s(\lambda,t)$$

$$\because F_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_s(\lambda,t) \text{ and } F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \lambda u(0,t) - \lambda^2\overline{u}_s(\lambda,t)$$

$$\Rightarrow \frac{d}{dt}\overline{u}_s(\lambda,t) = -k\lambda^2\overline{u}_s(\lambda,t) \qquad \because u(0,t) = 0$$

Rearranging the ordinary differential equation in variable separable form:

$$\Rightarrow \frac{d\overline{u}}{\overline{u}} = -k\lambda^2 dt \ldots ① \qquad \text{where } \overline{u} \approx \overline{u}_s(\lambda,t)$$

Solving ① using usual methods of variable separable differential equations

$$\log\overline{u} = -k\lambda^2 t + \log A$$

$$\Rightarrow \log\frac{\overline{u}}{A} = -k\lambda^2 t$$

$$\Rightarrow \overline{u}_s(\lambda,t) = A\, e^{-\mathrm{k}\lambda^2 t} \ldots \quad ②$$

Putting $t = 0$ on both sides

$$\Rightarrow \overline{u}_s(\lambda,0) = A \;\ldots \quad ③$$

Now given that $u(x,0) = e^{-x}$

Taking Fourier sine transform on both sides, we get

$$\Rightarrow \overline{u}_s\,(\lambda,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\lambda x\, dx$$

$$\Rightarrow \overline{u}_s(\lambda,0) = \sqrt{\frac{2}{\pi}}\,\frac{\lambda}{1+\lambda^2} \ldots \quad ④$$

From ③and ④, we get

$A = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} \ldots ⑤$

Using ⑤ in ②, we get

$\overline{u}_s(\lambda, t) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} e^{-k\lambda^2 t}$

Taking Inverse Fourier sine transform

$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{u}_s(\lambda, t) \sin \lambda x \, d\lambda$

$\Rightarrow u(x, t) = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{1+\lambda^2} e^{-k\lambda^2 t} \sin \lambda x \, d\lambda$

Example 29 The temperature $u(x, t)$ in a semi-infinite rod $0 < x < \infty$ is determined by the differential equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ subject to conditions:

i. $u = 0,$ when $t = 0$, $x \geq 0$
ii. $\frac{\partial u}{\partial x} = -k \ (a \ costant),$ when $x = 0, t > 0$

Solution: As range of x is $(0, \infty)$, and also value of $\left[\frac{\partial u}{\partial x}\right]_{x=0}$ is given in initial value conditions, applying Fourier cosine transform to both sides of the equation:

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = 2F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$\Rightarrow \frac{d}{dt}\overline{u}_c(\lambda, t) = -2\left[\frac{\partial u}{\partial x}\right]_{x=0} - 2\lambda^2 \overline{u}_c(\lambda, t)$

$\because F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_c(\lambda, t)$ and $F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - \lambda^2 \overline{u}_c(\lambda, t)$

$\Rightarrow \frac{d}{dt}\overline{u}_c(\lambda, t) = 2k - 2\lambda^2 \overline{u}_c(\lambda, t)$

$\Rightarrow \frac{d\overline{u}}{dt} + 2\lambda^2 \overline{u} = 2k \ldots ①$ where $\overline{u} \approx \overline{u}_c(\lambda, t)$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

where $P = 2\lambda^2$, $Q = 2k$

Integrating Factor (IF) = $e^{\int P dt} = e^{\int 2\lambda^2 dt} = e^{2\lambda^2 t}$

Solution of ① is given by

$\overline{u}.e^{2\lambda^2 t} = \int 2k.e^{2\lambda^2 t} dt + A$

$\Rightarrow \overline{u}.e^{2\lambda^2 t} = \frac{2ke^{2\lambda^2 t}}{2\lambda^2} + A$

$\Rightarrow \overline{u}_c(\lambda, t) = \frac{k}{\lambda^2} + Ae^{-2\lambda^2 t}$ ... ②

Putting $t = 0$ on both sides

$\Rightarrow \overline{u}_c(\lambda, 0) = \frac{k}{\lambda^2} + A$ ... ③

Now given that $u(x, 0) = 0$

Taking Fourier cosine transform on both sides, we get

$\Rightarrow \overline{u}_c(\lambda, 0) = \int_0^\infty u(x, 0) \cos \lambda x \, dx = 0$

$\Rightarrow \overline{u}_c(\lambda, 0) = 0$ ... ④

From ③and ④, we get

$A = -\frac{k}{\lambda^2}$ ... ⑤

Using ⑤ in ②, we get

$\overline{u}_c(\lambda, t) = \frac{k}{\lambda^2}\left(1 - e^{-2\lambda^2 t}\right)$

Taking Inverse Fourier cosine transform

$u(x, t) = \frac{2}{\pi}\int_0^\infty \overline{u}_c(\lambda, t) \cos \lambda x \, d\lambda$

$\Rightarrow u(x, t) = \frac{2k}{\pi}\int_0^\infty \left(\frac{1 - e^{-2\lambda^2 t}}{\lambda^2}\right) \cos \lambda x \, d\lambda$

Example 30 Using Fourier transforms, solve the equation $\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2}, x > 0,\ t > 0$ subject to conditions:

i. $y = \alpha,\ when\ x = 0,\ t > 0$
ii. $y = 0, when\ t = 0,\ x > 0$

Solution: As range of x is $(0, \infty)$, and also value of $y(0, t)$ is given in initial value conditions, applying Fourier sine transform to both sides of the given equation:

$F_s\left\{\frac{\partial y}{\partial t}\right\} = kF_s\left\{\frac{\partial^2 y}{\partial x^2}\right\}$

$\Rightarrow \frac{d}{dt}\overline{y}_s(\lambda, t) = k\lambda y(0, t) - k\lambda^2 \overline{y}_s(\lambda, t)$

$\because F_s\left\{\frac{\partial y}{\partial t}\right\} = \frac{d}{dt}\overline{y}_s(\lambda, t)$ and $F_s\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \lambda y(0, t) - \lambda^2 \overline{y}_s(\lambda, t)$

$\Rightarrow \frac{d}{dt}\overline{y}_s(\lambda,t) = k\alpha\lambda - k\lambda^2\overline{y}_s(\lambda,t)$ $\quad\because y(0,t) = \alpha$

$\Rightarrow \frac{d\overline{y}}{dt} + k\lambda^2\overline{y} = k\alpha\lambda$ ... ① $\quad$ where $\overline{y} \approx \overline{y}_s(\lambda,t)$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

where $P = k\lambda^2$, $Q = k\alpha\lambda$

Integrating Factor (IF) = $e^{\int Pdt} = e^{\int k\lambda^2 dt} = e^{k\lambda^2 t}$

Solution of ① is given by

$\overline{y}.e^{k\lambda^2 t} = \int k\alpha\lambda.e^{k\lambda^2 t}dt + A$

$\Rightarrow \overline{y}.e^{k\lambda^2 t} = \frac{k\alpha\lambda e^{k\lambda^2 t}}{k\lambda^2} + A$

$\Rightarrow \overline{y}_s(\lambda,t) = \frac{\alpha}{\lambda} + Ae^{-k\lambda^2 t}$ ... ②

Putting $t = 0$ on both sides

$\Rightarrow \overline{y}_c(\lambda,0) = \frac{\alpha}{\lambda} + A$ ... ③

Now given that $y(x,0) = 0$

Taking Fourier sine transform on both sides, we get

$\Rightarrow \overline{y}_s(\lambda,0) = \int_0^\infty y(x,0)\sin\lambda x\, dx = 0$

$\Rightarrow \overline{y}_s(\lambda,0) = 0$ ... ④

From ③and ④, we get

$A = -\frac{\alpha}{\lambda}$ ... ⑤

Using ⑤ in ②, we get

$\overline{y}_s(\lambda,t) = \frac{\alpha}{\lambda}\left(1 - e^{-k\lambda^2 t}\right)$

Taking Inverse Fourier sine transform

$y(x,t) = \frac{2}{\pi}\int_0^\infty \overline{y}_s(\lambda,t)\sin\lambda x\, d\lambda$

$\Rightarrow y(x,t) = \frac{2\alpha}{\pi}\int_0^\infty \left(\frac{1-e^{-k\lambda^2 t}}{\lambda}\right)\sin\lambda x\, d\lambda$

Example 31 An infinite string is initially at rest and its initial displacement is given by $f(x), -\infty < x < \infty$. Determine the displacement $y(x,t)$ of the string.

Solution: The equation of the vibrating string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Initial conditions are

i. $\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0$

ii. $y(x,0) = f(x)$

Taking Fourier transform on both sides

$$F\left\{\frac{\partial^2 y}{\partial t^2}\right\} = c^2 F\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d^2}{dt^2}\overline{y}(\lambda,t) = -c^2\lambda^2\,\overline{y}(\lambda,t) \quad \text{where } F\{y(x,t)\} \equiv \overline{y}(\lambda,t)$$

$$\Rightarrow \frac{d^2\overline{y}}{dt^2} + c^2\lambda^2\,\overline{y} = 0 \ldots ① \qquad \text{where } \overline{y} \approx \overline{y}(\lambda,t)$$

Solution of ① is given by

$\overline{y}(\lambda,t) = A\cos cpt + B\sin cpt$ ...②

Putting $t = 0$ on both sides

$\overline{y}(\lambda, 0) = A$ ... ③

Given that $y(x,0) = f(x)$

$\Rightarrow \overline{y}(\lambda, 0) = \overline{f}(\lambda)$ ... ④

From ③ and ④

$A = \overline{f}(\lambda)$ ... ⑤

Using ⑤ in ②

$\overline{y}(\lambda,t) = \overline{f}(\lambda)\cos cpt + B\sin cpt$ ...⑥

$\Rightarrow \frac{\partial y}{\partial t} = -cp\overline{f}(\lambda)\sin cpt + cpB\cos cpt$

$\Rightarrow \left.\frac{\partial y}{\partial t}\right]_{t=0} = cpB$ ... ⑦

Also given that $\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0$ ... ⑧

From ⑦ and ⑧, we get $B = 0$ ... ⑨

Using ⑨ in ⑥, we get

$\overline{y}(\lambda, t) = \overline{f}(\lambda) \cos cpt$

Taking inverse Fourier transform

$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{y}(\lambda, t) e^{-i\lambda x} d\lambda$

$\Rightarrow y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(\lambda) \cos cpt\, e^{-i\lambda x} d\lambda$

Exercise

1. Find the Fourier transform of $f(x) = \begin{cases} a - |x|, & |x| \le a \\ 0, & |x| \le a \end{cases}$

   Hence prove that $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

2. Solve the integral equation $\int_0^{\infty} f(x) \cos \lambda x\, dx = e^{-\lambda}, \lambda > 0$

3. Obtain Fourier sine integral of the function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

4. Prove that Fourier integral of the function $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & otherwise \end{cases}$ is given by

5. Find the Fourier sine and cosine transforms of xe^{-ax}

   $f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda.$ Hence show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

6. The temperature $u(x,t)$ at any point of a semi infinite bar satisfies the equation

   $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \; 0 < x < \infty, \; t > 0$, subject to conditions

   i. $u(0,t) = 0, \mathrm{t} > 0$

   ii. . $u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$ Determine the expression for $u(x,t)$

7. Determine the distribution of temperature in the semi infinite medium, $x \ge 0$, when the end at $x = 0$ is maintained at zero temperature and initial distribution of temperature is $f(x)$.

Answers

1. $\frac{2(1-\cos a\lambda)}{\lambda^2}$ 2. $f(x) = \frac{2}{\pi(1+x^2)}$ 3. $f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{(2\sin\lambda - \sin 2\lambda)}{\lambda^2} \sin \lambda x\, d\lambda$

5. $\frac{2a\lambda}{(a^2+\lambda^2)^2}, \frac{a^2-\lambda^2}{(a^2+\lambda^2)^2}$ 6. $u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos\lambda}{\lambda} e^{-\lambda^2 t} \sin \lambda x\, d\lambda$

7. $u(x,t) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_s(\lambda) e^{-c^2\lambda^2 t} \sin \lambda x\, d\lambda$

Unit –V NUMERICAL METHOD

Newton-Raphson method

Definition:

The Newton-Raphson formula is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \quad n=0, 1, 2\ldots\ldots\ldots$$

Rate of convergence:

The rate of convergence in Newton -Raphson method is order 2

Criterion for convergence:

(i) $f'(x_0)$ Should not be equal to zero. If $f'(x_0)=0$ then initial approximation must be changed.

(ii) For better convergence the product $f(x_0)f$(x} rsub {0} ¿¿ should be zero.

Problems:

1. What is a transcendental equation?

Equation which involves functions like logarithm, exponential, trigonometric etc is called transcendental equation.

(i) x+ Cos x+2=0 (ii) 2x + e^x-5=0

2. What is the rate of convergence in Newton - Raphson method?

The rate of convergence in Newton Raphson method is order 2

3 State the convergence condition for Newton Raphson method.

Condition for convergence is $|f(x)\ f''(x)| < ¿ f'(x) \vee ¿^2 ¿$

4. Find the first approximation of the root lying between 0 and 1 of the equation x^3+3x-1=0 by Newton - Raphson method.

f (x)= x^3+3x-1

f(0)= -1 (- ive)

f(1)=1+3-1=3 (+ ive)

so, a root lies between 0 and 1

Here |f (0)| >|f (1)|

Take $x_0=1$ f (1) =3

f' (X)=3x^2+3

f'(1)=3*1+3=6

$$x_1=x_0-\frac{f(x_0)}{f'(x¿¿0)¿}=1-\frac{3}{6}=0.5$$

5. Write the iterative formula of Newton -Raphson method?

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x¿¿n)¿}$, n=0, 1, 2..........

6. Write down Newton- Raphson formula for finding $\sqrt{a}$ where 'a' is a positive number?

$x_{n+1}=\frac{1}{2}¿+\frac{a}{x_n}]$

7. Write down newton raphson formula for finding 1/n where 'n' is a real number?

$x_{n+1}=x_n¿\mathbf{N}x_n¿$

8 . Find the first approximation of the equation $x\log_{10}x-1.2=0$ by newton raphson method correct correct to three decimal places?

Given, $x\log_{10}x-1.2=0$

Let f(x) $=x\log_{10}x-1.2$

f(1) $=1\log_{10}1-1.2$= -1.2= -ive

f(2) $=2\log_{10}2-1.2$= -0.598= -ive

f(3) $=3\log_{10}3-1.2$= 0.231= +ive

so , root lies between 2 and 3

Here |f(2)| >|f(3)|

Take $x_0=2.7$

Newton – Raphson formula is $x_{n+1}=x_n$-$\frac{f(x_n)}{f'(x¿¿n)¿}$, n=0, 1,

f'(x)= [x.$\frac{1}{X}$ $\log_{10}e$] $+\log_{10}X$

$=\log_{10}e$ $+\log_{10}X$

f(x_0) $=2.7\log_{10}2.7-1.2$= - 0.035

$f'(x_0)=¿$ $\log_{10}e$ $+\log_{10}2.7$=0.866

x_1=2.7—¿]=2.740

9. What is the criterion for the convergence in Newton Raphson method?

4 $f'(x¿¿0)¿$ Should not be equal to zero. If $f'(x¿¿0)¿=0$ then initial approximation must be changed.

5 For better convergence the product $f(x¿¿0)f(x\} rsub \{0\} ¿¿$ should be zero.

10 .Find the positive root of x^4-x=10 correct to three decimal places using Newton -Raphson method.

Solution:

Given x^4-x=100

Let f (x) =x^4-x-10

f (0) =0-0-10=-10(- ive)

f (1) = 1^4-1-10=-10(-ive)

f (2)= 2^4-2-10=4(+ive)

So, a root lies between 1 and 2

Here, |f (1)| >|f(2)

Therefore, the root is nearer to 2.

Let us take, x_0=2

The N.R formula is

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x¿¿n)¿}$,   n=0, 1,

f(x)= x^4-x-10 , $f'(x)=¿$ $4x^3$-1

n=0,x_0=2

f (x_0¿=2^4-2-10=16-2-10=4

$f'(x_0)=¿$32-1=31

x_1=2-$\frac{4}{31}$=1.871

n=1, x_1=1.871

f$(x¿¿1)¿$= (1.871^4¿-1.871-10

=0.383

$f'(x¿¿1)¿$= (4) (1.871^3¿-1

=25.199

x_2= x_1-$\frac{f(x¿¿1)}{f'(x¿¿1)¿}$¿

=1.871-$\frac{0.383}{25.199}$

x_2 =1.856

n=2, x=2=1.856

f (x_2¿=(($1.856¿¿^4$¿-1.856-10=0.010

$f'(x_2)$= (4) ($1.856¿¿^3$-1=24.574

$x_3=x_2-\frac{f(x¿¿2)}{f'(x¿¿2)¿}¿$

$=1.856-\frac{00.010}{24.574}$

=1.856.

Here $x_2=x_3$=1.856. Hence the better approximate rot is 1.856.

11. Using Newton's iterative method, find the root between 0 and 1 of x^3=6x-4 correct to 2 decimal places.

Given x^3=6x-4

$f(x)=x^3$-6x+4

f (0) =4 (+ive)

f (1) = -1=(-ive)

So, a root lies between 0 and 1

Here, |f (0)| > |f(1)

Therefore, the root is nearer to 1.

Let us take, x_0=1

The N.R formula is

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x¿¿n)¿}$, n=0, 1,

$f'(x)=¿\ 3x^2$-6

n=0,x_0=1

f (1¿=-1(- ive)

$f'(1) = ¿$ -3(- ive)

x_1=1-$\frac{-1}{3}$=0.67

n=1, x_1=0.67

f$(x¿¿1)¿$= ($0.67^3¿$-6(0.57)+4

=0.28

$f'(x¿¿1)¿$= 3($0.67^2¿$-6=-4.65

$x_2 = x_1$-$\frac{f(x¿¿1)}{f'(x¿¿1)¿}¿$

=0.67-$\frac{0.28}{-4.65}$

x_2 =0.73

n=2, x=2=0.73

f ($x_2¿$= ($0.73¿¿^3$-6(0.73)+4=0.01

$f'(x_2)$= (3)($0.73¿¿^2$-6=-4.40

$x_3 = x_2$-$\frac{f(x¿¿2)}{f'(x¿¿2)¿}¿$

=0.73-$\frac{0.01}{-4.40}$

=0.73

Here $x_2 = x_3$=0.73. Hence the better approximate rot is 0.73.

12. Find the real positive root of 3x-cosx-1=0 by Newton,s method correct to 6 decimal places.

Given , 3x-cosx-1=0

f(x)=3x-cosx-1

$f'(x)=3+\sin x$

f (0) =-2 (ive)

f (1) = 1.459698(+ive)

So, a root lies between 0 and 1

Here, |f (0)| >|f(1)

Therefore, the root is nearer to 0.

Let us take, $x_0=0.3$

The N.R formula is

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x¿¿n)¿}$, n=0, 1,

n=0,$x_0=0.6$

$f'(0.6)=¿$ 3.564642(+ ive)

$x_1=x_0-\frac{f(x¿¿0)}{f'(x¿¿0)¿}¿$

$x_1=0.6-\frac{0.025336}{3.564642}$

=0.607108

n=1, $x_1=0.607108$

$f(x¿¿1)¿=3(0.607108¿-\cos(0.607108)-1$

$=0.000023$ $x_2=$

$x_1-\frac{f(x¿¿1)}{f'(x¿¿1)¿}¿$

$=0.607108-\frac{0.000023}{3.570495}$

$x_2=0.607102$

Here $x_2=x_1$=0.607102. Hence the better approximate rot is 0.607102.

13.Solve by Newton's method ,a root of e^x-4x=0.

Given e^x-4x=0

f (x)=e^x-4x

f (0) =1 (+ive)

f (1) = -1.2817=(-ive)

So, a root lies between 0 and 1

Here, |f (0)| <|f(1)Therefore, the root is nearer to 0.

Let us take, x_0=0.3

The N.R formula is

$x_{n+1}=x_n-\frac{f(x_n)}{f'(x¿¿n)¿}$, n=0, 1,

$f'(x)=¿\ e^x-4$

$x_1=x_0-\frac{f(x_0)}{f'(x¿¿0)¿}$ =0.3-[$\frac{e^{0.3}-4(0.3)}{e^{0.3}-4}¿$

x_1=0.3 -$\frac{0.1499}{-2.650}$ =0.3566

x_2=x_1-$\frac{f(x¿¿1)}{f'(x¿¿1)¿}$¿=0.3566-[$\frac{e^{0.3566}-4(0.3566)}{e^{0.3566}-4}$

=0.3574

x_2 =0.3574

x_3=x_2-$\frac{f(x¿¿2)}{f'(x¿¿2)¿}$¿=0.3574-[$\frac{e^{0.3574}-4(0.3574)}{e^{0.3574}-4}$

=0.3574

Here x_2=x_3=0.3574. Hence the better approximate root is 0.3574

14 .Write down Newton Raphson formula for finding $\sqrt{a}$, where 'a' is a positive number and hence find $\sqrt{5}$

Let x=$\sqrt{a}$

x^2=a

x^2-a=0

Let f(x) = x^2-a

$f'(x)$ = 2 x

N-R formula is

x_{n+1}=x_n-$\frac{f(x_n)}{f'(x¿¿n)¿}$, n=0, 1,

$$= x_n - \frac{xn^2}{2x_n} + \frac{a}{2x_n}$$

$$¿ x_n - \frac{x_n}{2} + \frac{a}{2x_n}$$

$$= \frac{x_n}{2} + \frac{a}{2x_n}$$

$x_{n+1} = \frac{1}{2}[x_n + \frac{a}{2x_n}]$ is the iterative formula to Find $\sqrt{a}$.

To find $\sqrt{5}$

Put a = 5

Also x = $\sqrt{5}$ lies between 2 and 3

Let $x_0 = 2$.

$$x_{n+1} = \frac{1}{2}[x_n + \frac{a}{x_n}]$$

$$x_1 = \frac{1}{2}[x_0 + \frac{5}{x_0}]$$

$$= \frac{1}{2}[2 + \frac{5}{2}] = \frac{1}{2} = 2.25$$

$$x_2 = \frac{1}{2}[x_1 + \frac{5}{x_1}]$$

$$= \frac{1}{2}[2.25 + + \frac{5}{2.25}]$$

$$x_2 = 2.2361$$

$$x_3 ¿ \frac{1}{2}[x_2 + \frac{5}{x_2}]$$

$$= \frac{1}{2}[2.2361 + + \frac{5}{2.2361}]$$

$x_3 = 2.2361$

Here, $x_2 = x_3 = 2.2361$

Hence the approximate value of $\sqrt{5} = 2.2361$

15. Find the iterative formula for finding the value of 1/n where n is a real number using newton raphson method hence evaluate 1/26 correct to 4 decimal places

Let x = $\frac{1}{N}$

N = $\frac{1}{x}$

Let f(x) = $\frac{1}{x}$ - N,

f'(x) = - $\frac{1}{x^2}$

The N.R formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0, 1,$$

$$= x_n - \frac{\frac{1}{x_n} - N}{\frac{-1}{x_n^2}},$$

$$= x_n + x_n^2 \left[\frac{1}{x_n} - N\right]$$

$$= x_n + x_n - N.x_n^2$$

$$= 2x_n - N.x_n^2$$

$x_{n+1} = x_n [2 - N.x_n^2]$ is the iterative formula

To Find $\frac{1}{26}$, take N = 26.

Let x_0 = 0.04 [$x_0 = \frac{1}{25}$ = 0.04

$x_{n+1} = x_n$ [2 - Nx_n]

$x_1 = x_0$ [2 - 26x_0]

x_1 = (0.04) [2 – 26 (0.04)] = 0.0384

$x_2 = x_1$ [2 - 26x_1]

= (0.0384) [2 – 26 (0.0384)] = 0.0385

$x_3 = x_2$ [2- 26.x_2]

= (0.0385) [2 – 26 (0.0385)

x_3 = 0.0385

Hence the value of $\frac{1}{26}$ = 0.0385.

Trapezoidal rule:

Definition:

The Trapezoidal rule is

$$\int_{x_0}^{x_0++nh} f(x)\,dx = \frac{h}{2}[(y_0+y_n¿+ 2(y_1+y_2+....+y_{n-1})]$$

$=\frac{h}{2}$[(sum of the first and last term)+ 2 (sum of the remaining term)

Simpson's 1/3 rule:

Definition:

The Simpson's 1/3 rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0+y_n \text{¿} + 4(y_1+y_3+\ldots y_{n-1}) + \text{¿} + 2(y_2+y_4+\ldots.+y_{n-2}\text{¿}]$$

Simpson's 3/8th rule:

Definition

The Simpson's 3/8th rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0+y_n \text{¿} + 3(y_1+y_2+y_4+y_5\ldots.+y_{n-1}\text{¿} + 2(y_3+y_6+\ldots.+y_{n-3})$$

Problems:

16. What is the order of error in trapezoidal formula?

Error in the Trapezoidal formula is of the order h^2.

17. What is the order of error in Simpson's formula?

Error in the Simpson's formula is of the order h^4.

18. What is the error in trapezoidal rule of numerical integration ?

Error in trapezoidal rule is

$|E| < \frac{(b-a)}{12} h^2 M$ in the interval

(a, b), where $h = \frac{(b-a)}{n}$

19. What is the error in Simpson's rule of numerical integration?

$|E| < \frac{(b-a)}{180} h^4 M$ in the interval

20. Using trapezoidal rule, Evaluate $\int_0^{\Pi} \sin x \, dx$ by dividing the range into 6 equal parts.

x	0	$\frac{\Pi}{6}$	$\frac{2\Pi}{6}$	$\frac{3\Pi}{6}$	$\frac{4\Pi}{6}$	$\frac{5\Pi}{6}$	1
y	0	0.5	0.866	1	0.866	0.5	0

$$\int_0^{\Pi} \sin x \, dx = \int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{2}[(y_0+y_n ¿+2(y_1+y_2+....+y_{n-1})]$$

$$= \frac{\frac{\Pi}{6}}{2}[(0+0)+2(0.5+0.866+1+0.866+0.5]$$

$$=0.65136$$

21. Using Simpson's rule find $\int_0^4 e^x dx$

$given e^0=1, e^1=2.72, e^2=7.39, e^3=20.09, e^4=54.6$

Let f(x) $=e^x$

Take h=1

The Simpson's rule

$$\int_0^4 e x d\text{x} = \frac{h}{13} [(y_o+y_4 ¿ +2y_2+4 ¿+y_{3¿}¿]$$

$\frac{1}{3}$[(1+54.6) +2(7.39) +4(2.72+20.09)]

= 53.8733

22. Using trapezoidal rule evaluate $\int_{-1}^{1} \frac{1}{1+x^2}$ dx taking 8 intervals

Solution: Here y(x) = $\frac{1}{1+x^2}$

Length of the interval = 2 so, we divide 8 equal intervals with

h = $\frac{2}{8}$ = 0.25

By trapezoidal rule,

We get $\int_{-1}^{1} \frac{1}{1+x^2}$ dx =$\frac{h}{2}$ [(y_0 + y_h) + 2(y_1+y_2+…. + y_{n-1})]

= $\frac{0.25}{2}$ [(0.5+ 0.5) + 2(0.64+0.8+0.9412

1+0.8+0.64)]

= $\frac{0.25}{2}$ [1+2(5.7624)]

= $\frac{0.25}{2}$[12.5248] = 1.5656

23. Dividing the range into 10 equal parts, find the value of $\int_{0}^{\frac{\Pi}{2}} \sin x dx$ by (i) trapezoidal rule (ii) simpson's rule

Solution:

Given y(x) = $\sin x$, h= $\frac{\frac{\Pi}{2}}{10} = \frac{\Pi}{20}$

Divide the interval into 10 equal parts

X	0	$\frac{\Pi}{20}$	$\frac{2\Pi}{20}$	$\frac{3\Pi}{20}$	$\frac{4\Pi}{20}$
Y=$\sin x$	0	0.1564	03090	0.4540	0.5878

$\frac{5\Pi}{20}$	$\frac{6\Pi}{20}$	$\frac{7\Pi}{20}$	$\frac{8\Pi}{20}$	$\frac{9\Pi}{20}$	$\frac{10\Pi}{20}$
0.7071	0.8090	0.8910	0.9511	0.9877	1

(i) By trapezoidal rule

$$\int_0^{\frac{\Pi}{2}} \sin x dx = \frac{h}{2} [(y_o+y_n+21y_1+y_2+\ldots+y_n\text{-}1)]$$

$$= \frac{h}{2}[(y_o+y_{10}) + 2(y_1+y_2+\ldots. + y_9)]$$

$$= \frac{\frac{\Pi}{20}}{2}$$

[(0+1)+2(0.1564+0.3090+0.4540+0.5878+0.7071+0.8090

+0.8910+0.9511+0.9877)]

$$= \frac{\Pi}{40}[0+1+ 2(5.8531)]$$

$$= \frac{\Pi}{40}[0+1+12 .706$$

$$\int_0^{\frac{\Pi}{2}} \sin x dx$$

=0.9980

(ii)By Simpson's$\frac{1}{3}$ rule

$$\int_0^{\frac{\Pi}{2}} \sin x dx \frac{h}{3} [(y_o + y_n ¿ + 4 (y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{\frac{\Pi}{20}}{3} [(0+1) +(0.1564+0.4540+0.7071+0.8910+0.9877)$$

+2 (0.3090+0.5878+0.8090+0.9511)]

$$= \frac{\Pi}{60}[(0+1)+4(3.1962)+2(2.6569)$$

$$= \frac{\Pi}{60}[(1+12.7848+5.3138)]$$

$$= = \frac{\Pi}{60}[19.0986] =1.0000$$

24. Using Simpson's One third rule evaluate $\int_0^1 x e^x$ dx taking 4 intervals . Compare your result with actual value.

Solution:

Given f(x) = xe^x

Taking 4 intervals, h = $\frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$ =0.25

X	0	0.25	0.5	0.75	1
Y=xe^x	0	0.321	0.824	1.588	0.718

Simpson's$\frac{1}{3}$ rule is

$$\int_{xo}^{xn} f(x)\,dx = \frac{h}{3} \quad [(y_o+y_4) +4(y_1+y_3) +2(y_2)]$$

$$= \frac{0.25}{3} \ [(0+2.718)+4(0.321+1.588)+2(0.824)]$$

$$= \frac{0.25}{3} [2.718+7.636+1.648]$$

$$=\frac{0.25}{3} [12.002] =\frac{3.0005}{3} =1\ldots\ldots\ldots\ldots\ldots\ldots\ (A)$$

Actual value

$$\int_{o}^{1} x\,e^{x}dx = \int_{0}^{1} xd\,¿]$$

$$= [xe^{x}]_0^1 - \int_{o}^{1} e^{x}dx$$

$$= ¿-0] - [e^{x}]_0^1$$

$$= (e^{1} - 0) - [e^{1} - e^{0}]$$

$$=e^{1} - [e - 1]$$

$$=e - [e - 1] = 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots..B$$

Here A = B

So both the values are equal.

25. By dividing the range into ten equal parts, evaluate $\int_{0}^{\Pi} \sin x d x$ by trapezoidal and Simpson's rule. verify your answer with integration

Sol given, f(x) = $\sin x dx$

$$h = \frac{b-a}{n} = \frac{\Pi-0}{10} = \frac{\Pi}{10}$$

table value

X	0	$\frac{\Pi}{10}$	$\frac{2\Pi}{10}$	$\frac{3\Pi}{10}$	4Π
Y=$\sin x$	0	0.3090	0.5878	0.8090	0.9511

X	$\frac{5\Pi}{10}$	$\frac{6\Pi}{10}$	$\frac{7\Pi}{10}$	$\frac{8\Pi}{10}$	$\frac{9\Pi}{10}$	$\frac{\Pi}{10}$
Y=$\sin x$	1.0	0.9511	0.8090	0.5878	0.3090	0

(i)By trapezoidal rule, $\int_0^{\Pi} \sin x d x$

$$\int_0^{\Pi} \sin x d x = \frac{h}{2}[y_o + y_n) + 2(y_1 + y_2) + \ldots\ldots\ldots\ldots\ldots(y_{n-1})$$

$$= \frac{h}{2}[y_o + y_{10}) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ldots y_9)$$

$$= \frac{\frac{\Pi}{10}}{2}$$

[(0+0)+2(0.3090+0.5878+0.8090+0.9511+1.0+0.9511+0.8090+0.5878+0.3090]

=1.9843…………. (1)

(ii) Simpsons' rule

$$\int_0^{\Pi} \sin x d x = \frac{h}{3}[(y_0 + y_n ¿ + 4(y_1 + y_3 + \ldots y_{n-1}) + ¿ + 2(y_2 + y_4 + \ldots . + y_{n-2}$$

$= \frac{\frac{\Pi}{10}}{2}$

[(0+0)+4(0.3090+0.8090+1+0.8090+0.3090)+20.5878 +0.9511+0.9511+0.5878)]

$= \frac{\Pi}{30}$ *19.0996 =2.001………… (2)

(iii) Actual integration

$I=\int_0^{\Pi} \sin x dx =$ (-$\cos x ¿_0^{\Pi} ¿$

= -(cos Π-cos 0)

= (-1-1)

= 2……………………………(3)

Comparing (1), (2), and (3) Simpsons rule is more accurate that ha trapezoidal rule.

Euler's method :

Definition:

Euler's formula is

$y_{n+1} = y_n + hf(x_n, y_n)$**, n=0, 1, 2**

26. Solve $\frac{dy}{dx}$ =1-y, y(0)=0 for x=0.1 By euler's method

Given , f(x,y)=1-y, x=0, y=0 h=0.1

Euler's algorithm,

y_n+1= y_n+h f(x_n, y_n¿

$y_1 = y_0 + h\ f(x_0, y_0$¿¿¿

$=0+0.1(1-0)$

$y_1 = 0.1$

27. Using Euler's method find y (0.2) and y (0.4) from $\frac{dy}{dx} = x + y$, y (0)=1 with h=0.2

Solution:

Given f(x, y) = x + y

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.2$, $x_2 = 0.4$

Euler's Algorithm $y_{n+1} = y_n + h\ f(x_n, y_n$¿

$y_1 = y_0 + h\ f$ ¿ $y_0)$

$=1+0.2(0+1)$

$=1.2$

$y_2 = y_1 + h\ f$ ¿ $y_1)$

$=1.2+(0.2)(0.2+1.2)$

$=1.2+0.2(1.4)$

$=1.2+0.28=1.48$

$y_3 = y_2 + h\ f$ ¿ $y_2)$

= 1.48 +(0.2)(0.4+1.48)

=1.48+0.376

=1.856

28. Using Euler's Method , Solve $\frac{dy}{dx} = x + y$+xy, y (0)=1 with y(0)=1 compute y at x=0.1 by taking h=0.05.

Solution:

Given f(x, y) = x + y+ x y

$x_0 = 0$,$y_0 = 1$, h=0.05

Euler's Algorithm

y_{n+1}= y_n+h f (x_n, y_n¿

y_1= y_0+h f ¿ y_0)

=1+0.05[$x_0 + y_0$+x_0 y_0]

=1+0.05(0+1+0)

=1+0.05=1.05

y_2= y_1+h f ¿ y_1)

=1.05+0.05[$x_1 + y_1$+x_1 y_1]

=1.05+0.05[0.05+1.05+(0.05)(1.05)]

=1.05+0.05[1.1525]

$=1.05+0.057625$

$=1.107625$

29. Using Euler's Method , find the solution of the initial value problem .$\frac{dy}{dx}=\log ¿$, y (0)=2 at x=0.2 by assuming h=0.2.

Solution:

Given f(x, y) = $\log ¿$,

$x_0=0$,$y_0=2$, h=0.2

Euler's Algorithm

$y_{n+1}=\ y_n+h\ f\,(x_n,\ y_n ¿$

$y_1=y_0+h\ f ¿\ y_0)$

$=y_0+h\ \log ¿\ y_0)$

=2+0.2[log (0+2)]

=2+0.2 log 2

=2+0.2(0.3010)

y(0.2)=2.0602.

Runge - kutta method

Fourth order Runge - Kutta method for solving first order equations:

Properties:

(i) To evaluate y_{m+1}, they need only information at the point (x_m, y_m).

(ii) They don't involve the derivatives of f(x, y), such as in Taylor's series method.

(iii) They agree with the Taylor's series solution upto the terms of h^r, where r differs from method to method and is known as the order of that Runge - Kutta Method

Second order R-K method:

If the initial values of (x, y) for the differential equation $\frac{dy}{dx}$ =f(x, y) then the first increment in y namely Δy is calculated from the formula.

$$k_1 = \text{h f} ¿)$$

$$k_2 = \text{h f} ¿+\frac{h}{2}, y+\frac{k_1}{2}]$$

$$\Delta y = k_2 \text{ where } h = \Delta x.$$

$$k_1 = \text{h f} ¿)$$

$$k_2 = \text{h f} ¿+\frac{h}{2}, y+\frac{k_1}{2}]$$

k_3= h f$[x+h, y+2k_2-k_1]$

and $\Delta y=\frac{1}{6}(k_1+4k_2+k_3)$

k_1= h f ¿)

k_2= h f ¿+$\frac{h}{2}$, y+$\frac{k_1}{2}$]

k_3= h f ¿+$\frac{h}{2}$, y+$\frac{k2}{2}$) k_4= h f ¿x+h, y+k_3)

and $\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

y (x+ h)= y(x)+ Δy.

Working rule:

To solve $\frac{dy}{dx}= y^1$= f(x , y), y(¿)=y_0

k_1= h f ¿ y_0)

k_2= h f ¿+$\frac{h}{2}$, $y_0+\frac{k_1}{2}$) k_3= h f ¿+$\frac{h}{2}$, $y_0+\frac{k_2}{2}$) k_4= h f ¿+h, y_0+k_3)

and $\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$y_1=y_0+\Delta y$

where h=Δx

Now starting from ¿ y_1) and repeat the process.

.

30 Write the Runge- kutta algorithm of second order for solving y^1=f(x, y), y¿)=y_0

Let h denote the interval between equidistant values of x.

If the initial values are x_0, y_0) , the first increment in y is computed from the formulas

k_1= h f ¿ y_0)

k_2= h f ¿+$\frac{h}{2}$, y_0+$\frac{k1}{2}$,) and Δy= k_2 Thenx_1 = x_0 +h, y_1 = y_0+Δy

The increment is y in the second interval is computed in a similar manner using the same three formulas , using the values x ,y in the place of x_0, y_0 respectively

31. Write down the R-K formula of fourth order to solve $\frac{dy}{dx}$

=f(x, y) with y (x_0) = y_0

Let h denote the interval

If the initial values are ¿ y_0)

The first increment in y is computed from the formulas

k_1= h f ¿ y_0)

k_2= h f ¿+$\frac{h}{2}$, $y_0+\frac{k1}{2}$)

k_3= h f ¿+$\frac{h}{2}$, $y_0+\frac{k2}{2}$)

k_4= h f ¿+h, y_0+k_3)

and Δy=$\frac{1}{6}$ ($k_1+2k_2+2k_3+k_4$)

Then x_1 = x_0+h, $y_1=y_0$+Δy

The increment in y n the second interval is computed in a similar manner using the same four formulas, using the val*ue* x_1, y_1 in the place of x_0, y_0 respectively

32. Given $\frac{dy}{dx}$ = x^3+y, y (0)=2 compute y(0.2), y(0.4) by Runge-Kutta method of fourth order

Solution: Given $\frac{dy}{dx}$= y^1= x^3+y =f(x ,y)

x_0=0, y_0=2

x_1=¿0.2, x_2 =0.4, x_3=0.6

By fourth order R-K algorithm

k_1= h f¿ y_0)

k_2= h f ¿+$\frac{h}{2}$, y_0+$\frac{k1}{2}$) k_3= h f ¿+$\frac{h}{2}$, y_0+$\frac{k2}{2}$) k_4= h f ¿+h, y_0+k_3)

$$\Delta y=\frac{1}{6}\,(k_1+2k_2+2k_3+k_4)$$

y (x+ h)= y(x)+ Δy

(i) To find y(0.2)

y_1=0.2, x_0=0, y_0=2, h= 0.2

k_1= h f¿ y_0)

= (0.2) [x_0^3 + y_0]

= (0.2) [0+2]

= 0.2*2=0.4

k_2 =h f [x_0 +$\frac{h}{2}$,y_0+¿ $\frac{k1}{2}$]

= (0.2) f [0+¿ $\frac{0.2}{2}$, 2++¿ $\frac{0.4}{2}$]

= (0.2) f (0.1, 2.2)

= (0.2) [0.1^3+ 2.2]

= (0.2) (2.201)

= 0.4402

k_3 =h f [x_0 +$\frac{h}{2}$,y_0+¿ $\frac{k2}{2}$]

$= (0.2)\ f\,[0+¿\ \frac{0.2}{2},\ 2+¿\ \frac{0.4402}{2}]$

$= (0.2)\ f\,[\ 0.1\ ,\ 2.2201]$

$= (0.2)\ [0.1^3 + 2.2201]$

$= (0.2)\ (2.2211)$

$k_3 = 0.44422.$

$k_4 \quad = h\ f\,[x_0 + h, y_0 + k_3\ ]$

$= (0.2)\ f\,[0+0.2,\ 2+0.44422]$

$= (0.2)\ f\,[0.2,\ 2.44422]$

$= (0.2)\ [0.2^3 + 2.44422]$

$= (0.2)\ [2.44422]$

$k_4 = 0.44422.$

$\Delta y = \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0.4 + 2(0.4402) + 2(0.44422) + 0.490444)]$

$= \frac{1}{6}(2.65928)$

$= 0.44321$

$y(0.2) = 0.44321$

$y_1 = y_0 + \Delta y$

$= 2 + 0.44321 = 2.44321$

$y_1 = 2.44321$

(ii)To find y (0.4)

Apply R – K method

$k_1 = h\,f ¿\ y_1)$

$= 0.2\ f[0.2, 2.443]$

$= (0.2)\ [(0.2 ¿¿^3 + 2.443]$

$= (0.2)\ [2.451]$

$= 0.4902$

$k_2 = h\,f\left[x_1 + \frac{h}{2}, y_1 + ¿\ \frac{k1}{2}\right]$

$= (0.2)\ f\left[0.2 + ¿\ \frac{0.2}{2}, 2.443 + + ¿\ \frac{0.4902}{2}\right]$

$= (0.2)\ f\,(0.3, 2.6881)$

$= (0.2)\ [0.3^3 + 2.6881]$

$= (0.2)\ (2.7151)$

$= 0.5430$

$k_3 = h\,f\left[x_1 + \frac{h}{2}, y_1 + ¿\ \frac{k_2}{2}\right]$

$= (0.2)\ f\left[0.2 + ¿\ \frac{0.2}{2}, 2.443 + ¿\ \frac{0543}{2}\right]$

$= (0.2)\ f\,[0.3, 2.7145]$

$= (0.2)\ [¿ + 2.7145]$

$= (0.2)\ (2.7145)$

$k_3 = 0.5483.$

k_4 $=$h f [x_1 +h,y_1+k_3]

= (0.2) f [0.2+0.2, 2.4443+0.5483]

= (0.2) f [0.4, 2.9913]

= (0.2) [¿+ 2.9913)

= (0.2) (3.0553)

= 0.6111

Δy = $\frac{1}{6}$ $(k_1+2k_2+2k_3+k_4)$

= $\frac{1}{6}$[0.4902 + 2(0.543) + 2(0.5483) + 0.6111)]

= $\frac{1}{6}$(3.2839)

= 0.5473

y(0.4) =0.5473

$y_2=y_1+\Delta$y

= 2.443+0.5473=2.99

y_2=2.99

33. Using R-k method of fourth order solve $\frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}$ with

y (0) =1 at x=0.2.

Solution: Given $\frac{dy}{dx}=\frac{y^2-x^2}{y^2+x^2}$,

$x_0=0$, $y_0=1$

$x_1=$¿0.2, h=0.2

By fourth order R-K algorithm

k_1= h f¿ y_0)

k_2= h f ¿+$\frac{h}{2}$, y_0+$\frac{k1}{2}$)

k_3= h f ¿+$\frac{h}{2}$, y_0+$\frac{k2}{2}$)

k_4= h f ¿+h, y_0+k_3)

$$\Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

y (x+ h)= y(x)+ Δy

k_1= h f¿ y_0)

=0.2[$\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}$] = 0.2[$\frac{1-0}{1+0}$¿=0.2

k_2= h f ¿+$\frac{h}{2}$, y_0+$\frac{k1}{2}$)

= (0.2) f [0+$\frac{0.2}{2}$, 1+$\frac{0.2}{2}$]

= (0.2) f (0.1, 1.1)

= (0.2) [$\frac{1.2}{1.222}$]

=0.19672

k_3= h f ¿+$\frac{h}{2}$, y_0+$\frac{k2}{2}$)

= (0.2) f ¿+$\frac{0.2}{2}$, 1+$\frac{0.19672}{2}$)

= (0.2) f (0.1, 1.0983606)

=0.1967

k_4= h f ¿+h, y_0+k_3)

= (0.2) f (0.2, 1.1967)

= 0.1891

$\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=\frac{1}{6}$ (0.2+2(0.19672) + 2(0.1967) + 0.1891)

=0.19598

y (x+ h)= y(x)+ Δy

y (0.2)= y(x)+ Δy=y_0+Δy

= 1+0.19598=1.19598

34. Apply R-K method to find y(0.2) in steps of 0.1 if $\frac{dy}{dx}=x+y^2$ given that y(0)=1

Solution

$k_1=hf(x,y)$

$k_2=hf\left(x+\frac{h}{2},y+\frac{k_1}{2}\right)$

$k_2=hf\left(x+\frac{h}{2},y+\frac{k_1}{2}\right)$

$k_4=hf(x+h,y+k_3)$

$\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$y(x+h)=y(x)+\Delta y$

$k_1=0.1000$

$k_2=0.1152$

$k_3 = 0.1168$

$k_4 = 0.1347$

$\Delta y = 0.1165$

$y(0.1) = 1.1165$

<u>To find y(0.2)</u>

$k_1 = 0.1347$

$k_2 = 0.1551$

$k_3 = 0.1576$

$k_4 = 0.1823$

$\Delta y = 0.1571$

$y(0.2) = 1.2736$

35 Using R-K method to find y(1.2) and y(1.4) from $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ given that y(1)=0

Solution

$k_1 = hf(x, y)$

$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$

$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$

$k_4 = hf(x + h, y + k_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(x + h) = y(x) + \Delta y$

<u>To find y(1.2)</u>

$k_1 = 0.1462$

$k_2 = 0.1402$

$k_3 = 0.1399$

$k_4 = 0.148$

$\Delta y = 0.1348$

$y(1.2) = 0.1402$

<u>To find y(1.4)</u>

$k_1 = 0.1348$

$k_2 = 0.1303$

$k_3 = 0.1301$

$k_4 = 0.1260$

$\Delta y = 0.1303$

$y(0.2) = 0.2705$